Supplement

S.1 Coordinate system and conventions

Müller matrices describe the effect of optical elements on the Stokes vector with respect to a coordinate system and using a set of definitions about signs and directions. Different sets of definitions can be found in the literature as discussed in detail in Muller (1969); some of which are even inconsistent. The discussions led to the so-called *Muller (or Muller-Nebraska-) convention*, which we follow in this paper (see also Hauge et al. (1980)). We use a right-handed Cartesian coordinate system (see Fig. 7), in wich angles are defined counter-clock wise, i.e. from the x- to the y-axis, when looking against the z-axis. The local z-axis points in the propagation direction of the light. We define the reference coordinate system of the lidar setup by the orientation of the polarising beam-splitter (PBS) in the receiving optics. Light polarised with its E-vector on the x-axis, i.e. parallel to the incident plane of the PBS in Fig. 6, is mostly transmitted by a usual PBS, while light with polarisation in y-direction, i.e. perpendicularly polarised to the incident plane, is mostly reflected. The incident plane is spanned up by the direction of light propagation (z-axis, propagation vector ***k***) and the normal of the reflecting surface, which means that the incident plane in Fig. 6 is the x-z-plane). The parallel and perpendicular polarisations are also called the p- and s-polarisation, respectively.

The orientation of linearly polarised light is defined by the orientation of the plane of vibration, which contains both the electric vector E and the propagation vector ***k***.

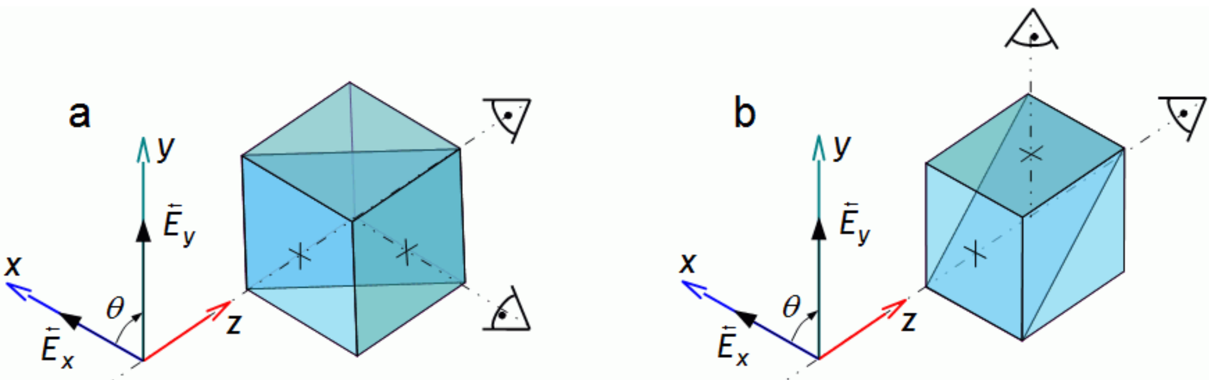

Fig. 6 Definition of the reference coordinate system with respect to the incidence plane of the polarising beam-splitter.

Other Müller matrix measurement configurations may have other arrangements for the coordinates. All choices, however, are arbitrary, and lead to different Müller matrices Chipman (2009b). There is no preferred set of definitions in the literature. According to our choice of orientation, the diattenuation parameter D ist defined as

$$D_T = \frac{T_T^p - T_T^s}{T_T^p + T_T^s},\ \Delta_T = \Delta_T^p - \Delta_T^s,$$

$$D_R = \frac{T_R^p - T_R^s}{T_R^p + T_R^s},\ \Delta_R = \Delta_R^p - \Delta_R^s$$

In order to keep the results of the Müller matrix calculations consistent when adding reflecting surfaces as mirrors and beam-splitters in the optical setup, a right-handed xyz-coordinate system is used with the z-axis in the direction of the light propagation. The vertical (perpendicular) polarised light has its E-vector in y-direction,

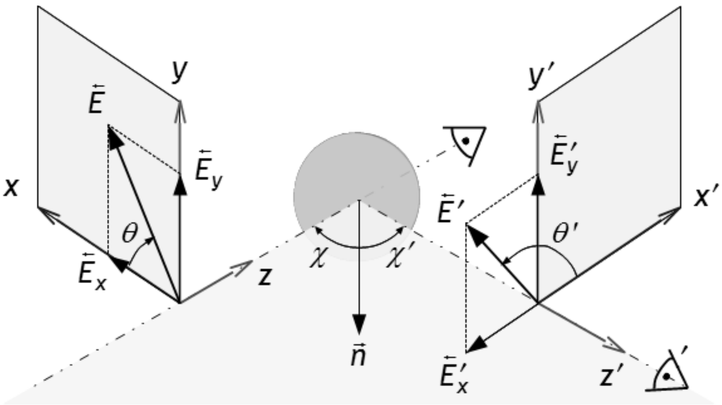

Fig. 7 Reflection of a Stokes vector.

S.2 Stokes vector and Müller matrix

The Stokes vector and the Müller matrix are one representation of the state of polarisation of light, which is a based on measurable quantities. The Stokes vector describes the polarisation state of a light beam, and the Müller matrix describes how the Stokes vector changes when passing through an optical volume, which can be an optical element or an atmospheric path with scattering, absorbing and refracting properties. A Stokes vector can be determined by six measurements of the flux I with ideal linear and circular polarisation analysers at different orientations before a detector Chipman (2009a; Ch. 15.17)

$$\begin{array}{ll} I^{p} & \text{parallel (horizontal) linear polarizer } (0°) \\ I^{s} & \text{perpendicular (vertical) linear polarizer } (90°) \\ I^{45} & 45° \text{ linear polarizer} \\ I^{135} & 135° \text{ linear polarizer} \\ I^{R} & \text{right ciruclar polarizer} \\ I^{L} & \text{left ciruclar polarizer} \end{array} \tag{S.2.1}$$

The Stokes vector is defined as

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I^{p} + I^{s} \\ I^{p} - I^{s} \\ I^{45} - I^{135} \\ I^{R} - I^{L} \end{pmatrix} \tag{S.2.2}$$

Right-circularly polarised light is defined as a clockwise rotation of the electric vector when the observer is looking against the direction of light propagation Bennett (2009a) (see Fig. X). Another representation, the so-called modified Stokes column vector Mishchenko et al. (2002) , uses the horizontally (parallel, *p*) and vertically (perpendicular, *s*) polarised fluxes I_p and I_s as the first two Stokes parameters. They can be transformed from the Stokes vector with a transformation matrix

$$\begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} = \mathbf{A} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} \Rightarrow \begin{array}{l} I_p = 0.5(I+Q) \\ I_s = 0.5(I-Q) \end{array} \tag{S.2.3}$$

and vice versa

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_p \\ I_s \\ U \\ V \end{pmatrix} \tag{S.2.4}$$

For fully polarised light, the Stokes parameters fulfil the equation

$$I^2 = Q^2 + U^2 + V^2 \tag{S.2.5}$$

and for full linearly polarised light

$$I^2 = Q^2 + U^2 \tag{S.2.6}$$

S.3 Depolarisation

Depolarisation is closely related to scattering and usually has it origins from retardance or diattenuation which is rapidly varying in time, wavelength, or spatially over an optical device (Cornu-, Lyot-, or wedge-depolariser). Depolarisation causes a loss of coherence of the polarisation state Chipman (2009a). The polarisation vector $\boldsymbol{I}_F$ reflected by the atmosphere $\mathbf{F}(a)$ with linear polarisation parameter a from a generally polarised laser $\boldsymbol{I}_L$ is (Sect. 5.32 of van de Hulst (1981); Sect. 4 of Mishchenko et al. (2002)).

$$\frac{\boldsymbol{I}_F(a)}{F_{11}T_E I_L} = \frac{\mathbf{F}(a)\left|\mathbf{M}_E \boldsymbol{I}_L\right\rangle}{F_{11}T_E I_L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \left| \begin{matrix} i_E \\ q_E \\ u_E \\ v_E \end{matrix} \right\rangle = \left| \begin{matrix} i_E \\ aq_E \\ -au_E \\ (1-2a)v_E \end{matrix} \right\rangle \quad \text{(S.3.1)}$$

The linear depolarisation ratio is defined as

$$\delta = \frac{F_{11} - F_{22}}{F_{11} + F_{22}} = \frac{1-a}{1+a} \Rightarrow a = \frac{1-\delta}{1+\delta} \quad \text{(S.3.2)}$$

With a linearly polarised laser with intensity I_L and linear polarisation parameter a_L and rotational misalignment α, i.e. without emitter optics, the laser light reflected by the atmosphere with linear polarisation parameter a is

$$\frac{\boldsymbol{I}_F(a,\alpha,a_L)}{F_{11}I_L} = \frac{\mathbf{F}(a)\left|\boldsymbol{I}_L(\alpha,a_L)\right\rangle}{F_{11}I_L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \left| \begin{matrix} 1 \\ a_L \mathrm{c}_{2\alpha} \\ a_L \mathrm{s}_{2\alpha} \\ 0 \end{matrix} \right\rangle = \left| \begin{matrix} 1 \\ aa_L \mathrm{c}_{2\alpha} \\ -aa_L \mathrm{s}_{2\alpha} \\ 0 \end{matrix} \right\rangle \quad \text{(S.3.3)}$$

It is obvious that the lasers a_L and the atmospheres a cannot be discerned in the resulting Stokes vector, and the measured, combined polarisation parameter is

$$a' = aa_L \quad \text{(S.3.4)}$$

The linear depolarisation ratio δ' resulting from a' can be retrieved with Eq. (12)

$$\delta' = \frac{1-a'}{1+a'} = \frac{\delta + \delta_L}{1 + \delta\delta_L} \quad \text{(S.3.5)}$$

For a small linear depolarisation ratio δ_L of the laser beam, the resulting linear depolarisation ratio of an atmospheric measurement is about the sum of the lasers and the atmospheres linear depolarisation ratios

$$\delta_L \ll 1 \Rightarrow \delta' \approx \delta + \delta_L \tag{S.3.6}$$

If δ_L is unknown, the uncertainty will cause an absolute error of the finally retrieved atmospheric linear depolarisation ratio.

S.4 Retarding linear diattenuator

The diattenuation magnitude D^* of an optical element, usually simply *diattenuation*, is calculated from the maximum and minimum transmitted intensities I (or transmittances T) (Chipman, 2009b), measured by rotating a linear polarising analyser in front of the element:

$$D^* = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{T_{\max} - T_{\min}}{T_{\max} + T_{\min}} \tag{S.4.1}$$

The diattenuation magnitude D^* is always positive, and if D^* is deduced. from the reflectances T_R^p and T_R^s of an optical sample as in Eq. (17), Eq. (S.4.1) becomes Lu and Chipman (1996)

$$D_R^* \equiv \left| \frac{T_R^p - T_R^s}{T_R^p + T_R^s} \right| \tag{S.4.2}$$

In order to avoid sign changes in the equations between the cases where $T_R^p < T_R^s$ and $T_R^p > T_R^s$, we use instead the diattenuation parameter D_S (Eq. (S.4.3)) (see Chipman (2009b), where it is named d_x or d_h), with which all equations can be expressed together for the transmitting (subscript $S = T$) and the reflecting (subscript $S = R$) part of a polarising beam-splitter.

$$D_S \in \{D_T, D_R\},\ D_R \equiv \frac{T_R^p - T_R^s}{T_R^p + T_R^s},\quad D_T \equiv \frac{T_T^p - T_T^s}{T_T^p + T_T^s} \tag{S.4.3}$$

The transmittances for unpolarised light are shown in Eq. (S.4.4), and some often occurring expressions in Eqs. (S.4.5) and (S.4.6).

$$T_R \equiv \frac{T_R^p + T_R^s}{2},\ T_T \equiv \frac{T_T^p + T_T^s}{2} \tag{S.4.4}$$

$$1 - D_R = T_R^s / T_R,\ 1 + D_R = T_R^p / T_R,\ 1 - D_T = T_T^s / T_T,\ 1 + D_T = T_T^p / T_T \tag{S.4.5}$$

$$\frac{T_R^s}{T_R^p} = \frac{1 - D_R}{1 + D_R},\ \frac{T_T^s}{T_T^p} = \frac{1 - D_T}{1 + D_T} \tag{S.4.6}$$

The optical elements considered here are non-depolarising, linear diattenuators $\mathbf{M}_D$, with linear diattenuation parameter D_O and average transmission T_O for unpolarised light, combined with linear retarders $\mathbf{M}_{Ret}$ (linear retardance $\varDelta_O$, $\cos\varDelta_O = \mathrm{c}_O$, $\sin\varDelta_O = \mathrm{s}_O$). The optical elements with possibly considerable diattenuation and retardation are dichroic beam-splitters, which are used to separate the wavelengths and to analyse the state of polarisation of the collimated beam in the receiver optics. They are used in transmission and reflection. The matrix of the transmitting part is Eq. (S.4.7) (see Eqs. (14, 15))

$$\mathbf{M}_T = \frac{1}{2}\begin{pmatrix} T_T^p + T_T^s & T_T^p - T_T^s & 0 & 0 \\ T_T^p - T_T^s & T_T^p + T_T^s & 0 & 0 \\ 0 & 0 & 2\sqrt{T_T^p T_T^s}\cos\varDelta_T & 2\sqrt{T_T^p T_T^s}\sin\varDelta_T \\ 0 & 0 & -2\sqrt{T_T^p T_T^s}\sin\varDelta_T & 2\sqrt{T_T^p T_T^s}\cos\varDelta_T \end{pmatrix} = \qquad \text{(S.4.7)}$$

$$= T_T\begin{pmatrix} 1 & D_T & 0 & 0 \\ D_T & 1 & 0 & 0 \\ 0 & 0 & Z_T \mathrm{c}_T & Z_T \mathrm{s}_T \\ 0 & 0 & -Z_T \mathrm{s}_T & Z_T \mathrm{c}_T \end{pmatrix}$$

$$Z_T \equiv \frac{2\sqrt{T_T^p T_T^s}}{T_T^p + T_T^s} = \sqrt{1 - D_T^2},\ \mathrm{c}_T \equiv \cos\varDelta_T,\ \mathrm{s}_T \equiv \sin\varDelta_T,\ \varDelta_T \equiv \varphi_T^p - \varphi_T^s \qquad \text{(S.4.8)}$$

with the shortcuts in Eq. (S.4.8), the intensity transmission coefficients (transmittance) for light polarised parallel (T_T^{p}) and perpendicular (T_T^{s}) to the plane of incidence of the PBS, the diattenuation parameter D_T (see S.3), and the average transmittance T_T for unpolarised light. $\varDelta_T$ is the difference of the phase shifts of the parallel and perpendicular polarised electrical fields. The Müller matrix for the reflecting part of the PBS (see Eqs. 16,17) includes a mirror reflection (S.6):

$$\mathbf{M}_R = T_R\begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & -Z_R \mathrm{c}_R & -Z_R \mathrm{s}_R \\ 0 & 0 & Z_R \mathrm{s}_R & -Z_R \mathrm{c}_R \end{pmatrix} = T_R\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & D_R & 0 & 0 \\ D_R & 1 & 0 & 0 \\ 0 & 0 & Z_R \mathrm{c}_R & Z_R \mathrm{s}_R \\ 0 & 0 & -Z_R \mathrm{s}_R & Z_R \mathrm{c}_R \end{pmatrix} \qquad \text{(S.4.9)}$$

with the corresponding intensity reflection coefficients (reflectance) for light polarised parallel (T_R^{p}) and perpendicular (T_R^{s}) to the plane of incidence of the PBS

$$Z_R \equiv \frac{2\sqrt{T_R^p T_R^s}}{T_R^p + T_R^s} = \sqrt{1 - D_R^2},\ \mathrm{c}_R \equiv \cos\varDelta_R,\ \mathrm{s}_R \equiv \sin\varDelta_R,\ \varDelta_R \equiv \varphi_R^p - \varphi_R^s \qquad \text{(S.4.10)}$$

In order to simplify the derivation of the equations, we write in the following for both, the reflecting and transmitting matrix of the polarising beam-splitter, $\mathbf{M}_S$ (subscript S for splitter) where appropriate.

$$Z_S \in \{-Z_R, Z_T\},\ \mathbf{M}_S \in \{\mathbf{M}_R, \mathbf{M}_T\},\ I_S \in \{I_R, I_T\} \tag{S.4.11}$$

eigen-polarisations along the x and y axes

$$\mathbf{M}_O = \mathbf{M}_D \mathbf{M}_{ret} = T_O \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O & 0 \\ 0 & 0 & 0 & Z_O \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{c}_O & \mathrm{s}_O \\ 0 & 0 & -\mathrm{s}_O & \mathrm{c}_O \end{pmatrix} = T_O \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O \mathrm{c}_O & Z_O \mathrm{s}_O \\ 0 & 0 & -Z_O \mathrm{s}_O & Z_O \mathrm{c}_O \end{pmatrix} \tag{S.4.12}$$

As both are linear, they commute (Eq. (S.4.13). They have a block-diagonal structure.

$$\mathbf{M}_O = \mathbf{M}_D \mathbf{M}_{ret} = \mathbf{M}_{ret} \mathbf{M}_D \tag{S.4.13}$$

Among such optical elements are λ/4 plates (Sect. S.10.16), λ/2 plates (Sect. S.10.13), dichroic beam-splitters, polarising beam-splitters, polarising sheet filters , aluminium and dielectric mirrors, and also uncoated glass surfaces under oblique incident angles. For further information see Azzam (2009); Bennett (2009a,b); Chipman (2009b,a)

S.5 Rotation

S.5.1 Rotation about the direction of light propagation

Some confusion can arise because rotation about the optical axis can be done on a Stokes vector, on the coordinate system (coordinate transformation), and on an optical element while keeping the reference coordinate system. The first two rotations don't change the state of the circular polarisation, while a rotated optical element can do that. Additional confusion arises because often in the literature and in textbooks the vector and coordinate rotations are mixed, or the derivation of the presented final equations from first principles are not provided, and sometimes the explanatory text is misleading or inconsistent. We follow the notations in Mishchenko et al. (2002); Chipman (2009b). Rotations are anti-clockwise, from the x-axis towards the y-axis, seen against the direction of light propagation (z-axis).

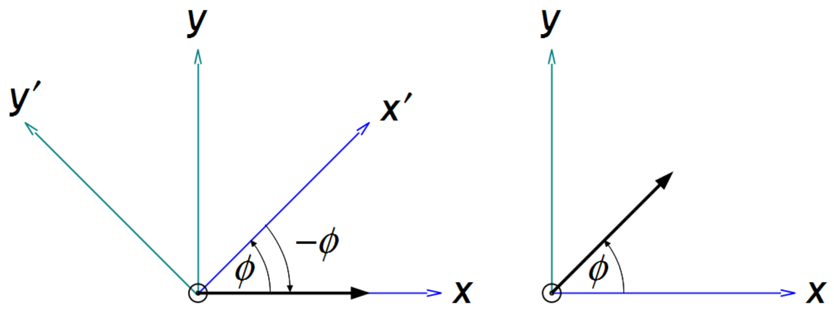

Fig. 8 Rotation of the xy-coordinate system (left) and of a vector (right). The z-axis, i.e. the direction of light propagation, points out of the paper plane.

A Stokes vector which is physically rotated by an angle ϕ while the coordinate system is fixed becomes Mishchenko et al. (2002; Ch. 1.5)

$$\boldsymbol{I}(\phi) = \mathbf{R}(\phi)\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ c_{2\phi} Q - s_{2\phi} U \\ s_{2\phi} Q + c_{2\phi} U \\ V \end{pmatrix} \tag{S.5.1.1}$$

with the rotation matrix $\mathbf{R}(\alpha)$ and the abbreviations

$$c_{2\phi} \equiv \cos 2\phi, \quad s_{2\phi} \equiv \sin 2\phi\,, \tag{S.5.1.2}$$

and

$$\mathbf{R}(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2\phi} & -s_{2\phi} & 0 \\ 0 & s_{2\phi} & c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S.5.1.3}$$

$$\mathbf{R}(90^\circ + \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -c_{2\phi} & s_{2\phi} & 0 \\ 0 & -s_{2\phi} & -c_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S.5.1.4}$$

$$\mathbf{R}(\pm 45^\circ + \varepsilon) = \mathbf{R}(\mathrm{x}45^\circ + \varepsilon) = \mathbf{R}(\mathrm{x},\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{x}s_{2\varepsilon} & -\mathrm{x}c_{2\varepsilon} & 0 \\ 0 & \mathrm{x}c_{2\varepsilon} & -\mathrm{x}s_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{S.5.1.5}$$

With these definitions a formula for one rotation can easily be converted to other angles with

$$
\begin{matrix}
\Psi & \rightarrow & 0^\circ+\varepsilon & \leftrightarrow & \pm 45^\circ+\varepsilon \\
\mathbf{R}(\Psi) & \rightarrow & \mathbf{R}(\varepsilon) & \leftrightarrow & \mathbf{R}(\mathrm{x}45^\circ+\varepsilon) \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{2\Psi} & -\mathrm{s}_{2\Psi} & 0 \\ 0 & \mathrm{s}_{2\Psi} & \mathrm{c}_{2\Psi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \rightarrow & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{2\varepsilon} & -\mathrm{s}_{2\varepsilon} & 0 \\ 0 & \mathrm{s}_{2\varepsilon} & \mathrm{c}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \leftrightarrow & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{xs}_{2\varepsilon} & -\mathrm{xc}_{2\varepsilon} & 0 \\ 0 & \mathrm{xc}_{2\varepsilon} & -\mathrm{xs}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\mathrm{c}_{2\Psi} & \rightarrow & \mathrm{c}_{2\varepsilon} & \leftrightarrow & -\mathrm{xs}_{2\varepsilon} \\
\mathrm{s}_{2\Psi} & \rightarrow & \mathrm{s}_{2\varepsilon} & \leftrightarrow & \mathrm{xc}_{2\varepsilon}
\end{matrix}
\tag{S.5.1.6}
$$

Please note, that in Mishchenko et al. (2002; Ch. 1.5) the equations describe a rotation of the Stokes vector, while the text specifies the transformation as "rotation of the two-dimensional coordinate system". The two transformations are called "alibi" and "alias" transformation Steinborn and Ruedenberg (1973), respectively. The Stokes vector rotates contra-variantly under the change of basis. If we rotate the coordinate system (alias transformation) by an angle ϕ (see Fig. 8), the original Stokes vector $\boldsymbol{I}$ appears in the rotated coordinate system under the angle $-\phi$, and the Stokes vector $\boldsymbol{I}'$ in the rotated coordinate system is Eq. (S.5.1.7).

$$
\boldsymbol{I}' = \begin{pmatrix} I' \\ Q' \\ U' \\ V' \end{pmatrix} = \mathbf{R}(-\phi)\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{2\phi} & \mathrm{s}_{2\phi} & 0 \\ 0 & -\mathrm{s}_{2\phi} & \mathrm{c}_{2\phi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ \mathrm{c}_{2\phi} Q + \mathrm{s}_{2\phi} U \\ -\mathrm{s}_{2\phi} Q + \mathrm{c}_{2\phi} U \\ V \end{pmatrix}
\tag{S.5.1.7}
$$

The rotation of the polarisation of a Stokes vector can be accomplished by means of a λ/2 plate (HWP), which is a 180° linear retarder. An ideal HWP can be derived from Eq. (S.5.2.3) by setting $\Delta_0 = 180°$, $D_{\grave{O}} = 0 \Rightarrow Z_O = 1$ and $W_O = 2$, and $T_O = 1$:

$$
\begin{aligned}
\mathbf{M}_{HWP}(\phi) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\mathrm{s}_{2\phi}^{\;2} & 2\mathrm{s}_{2\phi}\mathrm{c}_{2\phi} & 0 \\ 0 & 2\mathrm{s}_{2\phi}\mathrm{c}_{2\phi} & 1-2\mathrm{c}_{2\phi}^{\;2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{4\phi} & \mathrm{s}_{4\phi} & 0 \\ 0 & \mathrm{s}_{4\phi} & -\mathrm{c}_{4\phi} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\
&= \mathbf{R}(2\phi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
\tag{S.5.1.8}
$$

A HWP rotates a Stokes vector by twice the own rotation and additionally changes the direction of the circularly polarised component. For a rotation of $\phi = 90°$ the HWP acts as a mirror but without changing the direction of light propagation. Real HWPs are often made of birefringent crystals. Their retardance depends in general on the wavelength, on the incident angle, and on the temperature. For lidar applications so-called true zero-order HWPs are best suited. The HWP rotator and the mechanical rotator can be combined in one matrix $\mathbf{M}_{rot}$ as shown in S.10.15.

S.5.2 Rotation of a retarding diattenuator

The rotation of an optical element with Müller matrix $\mathbf{M}_O$ by an angle ϕ about the direction of light propagation is mathematically performed by first rotating the coordinate system before $\mathbf{M}_O$ by $-\phi$, to achieve the description of the Stokes vector in the local coordinate system (eigen-polarisations) of $\mathbf{M}_O$, and then rotating the coordinate system behind $\mathbf{M}_O$ back to the reference coordinate system by ϕ using the rotation matrix $\mathbf{R}(\phi)$

$$\mathbf{M}_O(\phi) = \mathbf{R}(\phi)\mathbf{M}_O(0°)\mathbf{R}(-\phi), \tag{S.5.2.1}$$

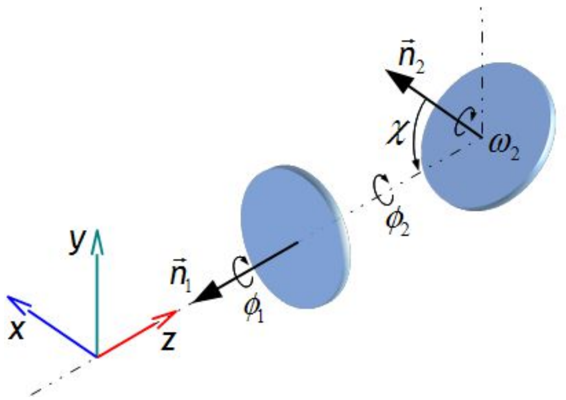

Figure 9 Rotation angles of an optical element. The rotations considered in this work are only ϕ_1 and ϕ_2.

A linear retarding diattenuator $\mathbf{M}_O$ rotated by ϕ about the z-axis becomes

$$\mathbf{M}_O(\phi)=\mathbf{R}(\phi)\mathbf{M}_O\mathbf{R}(-\phi)=$$

$$=T_O\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{s}_{2\phi}&0\\0&\mathrm{s}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&Z_O\mathrm{c}_O&Z_O\mathrm{s}_O\\0&0&-Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&\mathrm{s}_{2\phi}&0\\0&-\mathrm{s}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}=$$

$$=T_O\begin{pmatrix}1&\mathrm{c}_{2\phi}D_O&\mathrm{s}_{2\phi}D_O&0\\\mathrm{c}_{2\phi}D_O&1-\mathrm{s}_{2\phi}{}^2W_O&\mathrm{s}_{2\phi}\mathrm{c}_{2\phi}W_O&-\mathrm{s}_{2\phi}Z_O\mathrm{s}_O\\\mathrm{s}_{2\phi}D_O&\mathrm{s}_{2\phi}\mathrm{c}_{2\phi}W_O&1-\mathrm{c}_{2\phi}{}^2W_O&\mathrm{c}_{2\phi}Z_O\mathrm{s}_O\\0&\mathrm{s}_{2\phi}Z_O\mathrm{s}_O&-\mathrm{c}_{2\phi}Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix} \qquad \text{(S.5.2.2)}$$

$$\mathrm{c}_{2\phi}=\cos 2\phi,\mathrm{s}_{2\phi}=\sin 2\phi,\mathrm{c}_O=\cos\Delta_O,\mathrm{s}_O=\sin\Delta_O,\ \ Z_O\equiv\sqrt{1-D_O{}^2},\ \ W_O=1-Z_O\mathrm{c}_O \qquad \text{(S.5.2.3)}$$

Without diattenuation we get

$$D_O=0\Rightarrow\ Z_O=\sqrt{1-D_O{}^2}=1,\ \ W_O=1-\mathrm{c}_O \qquad \text{(S.5.2.4)}$$

and with ideal diattenuation

$$|D_O|=1,\ \ Z_O=\sqrt{1-D_O{}^2}=0,\ \ W_O=1 \qquad \text{(S.5.2.5)}$$

Rotation of a retarding diattenuator by ±45° + ε

$$\mathbf{M}_O(\mathrm{x}45^\circ+\varepsilon)=\mathbf{R}(\mathrm{x}45^\circ)\mathbf{M}_O(\varepsilon)\mathbf{R}(-\mathrm{x}45^\circ)=$$

$$=T_O\begin{pmatrix}1&0&0&0\\0&0&\mathrm{x}&0\\0&-\mathrm{x}&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&\mathrm{c}_{2\varepsilon}D_O&\mathrm{s}_{2\varepsilon}D_O&0\\\mathrm{c}_{2\varepsilon}D_O&1-\mathrm{s}_{2\varepsilon}^2W_O&\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&-\mathrm{s}_{2\varepsilon}Z_O\mathrm{s}_O\\\mathrm{s}_{2\varepsilon}D_O&\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&1-\mathrm{c}_{2\varepsilon}^2W_O&\mathrm{c}_{2\varepsilon}Z_O\mathrm{s}_O\\0&\mathrm{s}_{2\varepsilon}Z_O\mathrm{s}_O&-\mathrm{c}_{2\varepsilon}Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&-\mathrm{x}&0\\0&\mathrm{x}&0&0\\0&0&0&1\end{pmatrix}=$$

$$=T_O\begin{pmatrix}1&\mathrm{x}\mathrm{s}_{2\varepsilon}D_O&-\mathrm{x}\mathrm{c}_{2\varepsilon}D_O&0\\\mathrm{x}\mathrm{s}_{2\varepsilon}D_O&1-\mathrm{c}_{2\varepsilon}^2W_O&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&\mathrm{x}\mathrm{c}_{2\varepsilon}Z_O\mathrm{s}_O\\-\mathrm{x}\mathrm{c}_{2\varepsilon}D_O&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&1-\mathrm{s}_{2\varepsilon}^2W_O&\mathrm{x}\mathrm{s}_{2\varepsilon}Z_O\mathrm{s}_O\\0&-\mathrm{x}\mathrm{c}_{2\varepsilon}Z_O\mathrm{s}_O&-\mathrm{x}\mathrm{s}_{2\varepsilon}Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}=\mathbf{R}(\varepsilon)\mathbf{M}_O(\mathrm{x}45^\circ)\mathbf{R}(-\varepsilon) \qquad \text{(S.5.2.6)}$$

Without error angle ε:

$$\mathbf{M}_O(\mathrm{x}45^\circ)=\mathbf{R}(\mathrm{x}45^\circ)\mathbf{M}_O\mathbf{R}(-\mathrm{x}45^\circ)=X_O\begin{pmatrix}1&0&\mathrm{x}D_O&0\\0&Z_O\mathrm{c}_O&0&-\mathrm{x}Z_O\mathrm{s}_O\\\mathrm{x}D_O&0&1&0\\0&\mathrm{x}Z_O\mathrm{s}_O&0&Z_O\mathrm{c}_O\end{pmatrix} \qquad \text{(S.5.2.7)}$$

S.6 Mirror

For a pure mirror without diattenuation or retardance the Müller matrix is

$$\mathbf{M}_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{S.6.1}$$

which results from the rotation of the detector (the eye) about the y-axis from the rear of the optical element to the front as shown in Fig. 10.

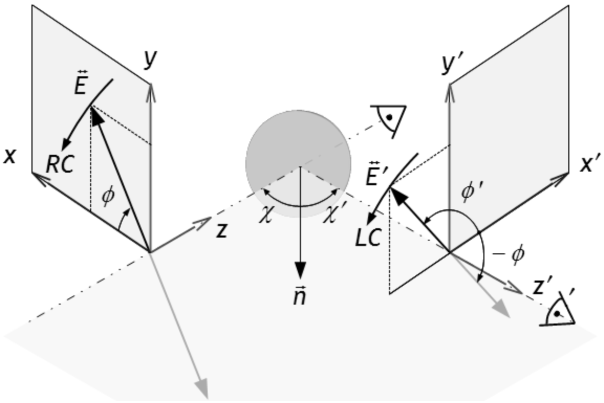

Figure 10: Reflection of light by a mirror. The light propagation is along the z-axis. The plane of vibration of linearly polarised light is indicated by the E-vectors, and right and left circular polarised light by the RC and LC arrows, respectively.

To explain the change of the axes, let the plane of vibration of linearly polarised light be rotated in the (xyz) coordinate system by ϕ around the z-axis, indicated by the E -vector in Fig. 10, and the incident angle be $\chi = 0$ for reflection from a mirror. After the mirror the direction of light propagation has changed, but not the orientation of the plane of vibration, indicated by the E'-vector. Hence, the rotation ϕ' in the mirrored coordinate system (xyz)' is $\phi' = 180° - \phi$, which is equivalent to $\phi' = -\phi$. Thus a Stokes vector rotated by $\mathbf{R}(\phi)$ in (xyz) is described in (xyz)' after the mirror $\mathbf{M}_M$ by

$$\boldsymbol{I} = \mathbf{M}_M \mathbf{R}(\phi) \boldsymbol{I} = \mathbf{R}(-\phi) \mathbf{M}_M \boldsymbol{I} \tag{S.6.2}$$

Furthermore, the circular polarisation has changed its sign from right circular (RC) before to left circular (LC) after the mirror.

S.6.1 Real mirror

Real mirrors are dielectric or metal surfaces which can exhibit considerable phase retardation and diattenuation under oblique incident angles. Hence, a real mirror is a linear retarding diattenuator $\mathbf{M}_O$ combined with a mirror $\mathbf{M}_M$.

$$\mathbf{M}_{MO} = = T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & Z_O \mathrm{c}_O & Z_O \mathrm{s}_O \\ 0 & 0 & -Z_O \mathrm{s}_O & Z_O \mathrm{c}_O \end{pmatrix} = T_O \begin{pmatrix} 1 & D_O & 0 & 0 \\ D_O & 1 & 0 & 0 \\ 0 & 0 & -Z_O \mathrm{c}_O & -Z_O \mathrm{s}_O \\ 0 & 0 & Z_O \mathrm{s}_O & -Z_O \mathrm{c}_O \end{pmatrix} \quad \text{(S.6.1.1)}$$

which commute

$$\mathbf{M}_{MO} = \mathbf{M}_M \mathbf{M}_O = \mathbf{M}_O \mathbf{M}_M = \mathbf{M}_{OM} \quad \text{(S.6.1.2)}$$

Eq. (S.6.1.1) is also the description of the reflecting part of a polarising beam-splitter or of any dichroic beam-splitter.

S.6.2 Rotation of a reflecting surface

If we rotate $\mathbf{M}_{MO}$, we have to mind the change of the coordinate system after the mirror. Here it is important which element comes first, because, as explained above, applying a mirror means a change of the local coordinate system after the mirror, and rotation of elements are always done with respect to the local coordinate system before the element. Hence, a diattenuator rotated in (xyz) plus a mirror described in (xyz)' is using (S.5.1.5) and (S.6.2)

$$\mathbf{M}_M \mathbf{M}_O(\phi) = \mathbf{M}_M \mathbf{R}(\phi) \mathbf{M}_O \mathbf{R}(-\phi) = \mathbf{R}(-\phi) \mathbf{M}_M \mathbf{M}_O \mathbf{R}(-\phi) = \mathbf{R}(-\phi) \mathbf{M}_{MO} \mathbf{R}(-\phi) = = \mathbf{M}_{MO}(\phi) \quad \text{(S.6.2.1)}$$

Moving the mirror before the diattenuator

$$\mathbf{M}_O(\phi) \mathbf{M}_M = \mathbf{R}(\phi) \mathbf{M}_O \mathbf{R}(-\phi) \mathbf{M}_M = \mathbf{R}(\phi) \mathbf{M}_O \mathbf{M}_M \mathbf{R}(\phi) = \mathbf{R}(\phi) \mathbf{M}_{OM} \mathbf{R}(\phi) = \mathbf{M}_{MO}(-\phi) \quad \text{(S.6.2.2)}$$

we see from (S.6.1.2) to (S.6.2.2) that

$$\mathbf{M}_O(\phi) \mathbf{M}_M = \mathbf{M}_{OM}(\phi) = \mathbf{M}_M \mathbf{M}_O(-\phi) = \mathbf{M}_{MO}(-\phi). \quad \text{(S.6.2.3)}$$

This explains why the rotation of a reflecting diattenuator has to be described as shown by Chipman (2009b) , i.e.:

$$\mathbf{M}_{OM}(\phi) = \mathbf{R}(\phi)\mathbf{M}_{OM}\mathbf{R}(\phi). \qquad \text{(S.6.2.4)}$$

S.6.3 Beam-splitters and mirrors in the optical path

In order to make the equations developed in this work applicable to a variety of lidar systems, we have to investigate how the equations are changed when individual elements are changed from transmitting to reflecting. This is also useful when the reflected and the transmitted paths after a beam-splitter are to be described with the same equations.

Above we showed the local coordinate change behind a mirror. But how does this effect the outcome of a lidar measurement and of the calibration measurements? Let's consider a chain of rotated optical elements using the eigen-polarisations of the polarising beam-splitter matrix $\mathbf{M}_S$ as the reference coordinate system

$$\boldsymbol{I}_S = \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(\phi)\mathbf{M}_1(\varepsilon)\mathbf{F}\boldsymbol{I}_{in} \qquad \text{(S.6.3.1)}$$

When we exchange $\mathbf{M}_2$ with its reflecting counterpart $\mathbf{M}_M \mathbf{M}_2$, we can move the ideal mirror $\mathbf{M}_M$ step by step to the right in the chain using (S.6.2.3)

$$\begin{aligned} \boldsymbol{I}'_S &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_M\mathbf{M}_2(\phi)\mathbf{M}_1(\varepsilon)\mathbf{F}\boldsymbol{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(-\phi)\mathbf{M}_M\mathbf{M}_1(\varepsilon)\mathbf{F}\boldsymbol{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(-\phi)\mathbf{M}_1(-\varepsilon)\mathbf{M}_M\mathbf{F}\boldsymbol{I}_L(\alpha) = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(-\phi)\mathbf{M}_1(-\varepsilon)\mathbf{F}\boldsymbol{I}^*{}_L(-\alpha) \end{aligned} \qquad \text{(S.6.3.2)}$$

and see that all rotation angles before the changed element are inversed. In the last step of Eq. (S.6.3.2) the depolarising atmospheric **F**-matrix is rotational invariant, and the circular polarisation of the input Stokes vector changes its sign, indicated by the star.

In other words: equations, which are derived for the system in Eq. (S.6.3.1), can be used for the system with an additional mirror as in Eq. (S.6.3.2) by inverting in the original equations all rotation angles before the mirror and reversing the circular polarisation of the input Stokes vector. In case two surfaces are changed from transmitting to reflecting as

$$\begin{aligned} \boldsymbol{I}''_S &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_M\mathbf{M}_2(\phi)\mathbf{M}_1(\varepsilon)\mathbf{F}\mathbf{M}_M\boldsymbol{I}_{in} = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(-\phi)\mathbf{M}_1(-\varepsilon)\mathbf{M}_M\mathbf{M}_M\mathbf{F}\boldsymbol{I}_{in} = \\ &= \eta_S \mathbf{M}_S \mathbf{M}_3(\gamma)\mathbf{M}_2(-\phi)\mathbf{M}_1(-\varepsilon)\mathbf{F}\boldsymbol{I}_{in} \end{aligned} \qquad \text{(S.6.3.3)}$$

where a mirror is additionally placed behind the emitter optics, only the rotation angles between these two elements are inversed, because $\mathbf{M}_M\mathbf{M}_M = \mathbf{1}$, and the circular polarisation is not changed.

Real mirrors are usually dielectric or metal surfaces which can exhibit considerable phase retardation and diattenuation under oblique incident angles. For incident angles smaller than the Brewster angle the phase changes for p- (parallel) and s- (perpendicular) polarised light are "in the same direction".

S.7 Standard atmospheric measurement signals

S.7.1 Lidar signal with rotational error before the polarising beam-splitter

General case with arbitrary laser input and emitter optics $\boldsymbol{I}_E = \mathbf{M}_E \boldsymbol{I}_L$ and $\mathbf{R}(\varepsilon,\mathrm{h})$ from Eq. (S.10.15.1):

$$\left\langle \mathbf{A}_S \right| = \left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \right| \mathbf{R}(\varepsilon)\mathbf{M}_\mathrm{h} = T_S \left\langle 1 \quad \mathrm{y}D_S \quad 0 \quad 0 \right| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{2\varepsilon} & -\mathrm{hs}_{2\varepsilon} & 0 \\ 0 & \mathrm{s}_{2\varepsilon} & \mathrm{hc}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & \mathrm{h} \end{pmatrix} = \tag{S.7.1.1}$$

$$= T_S \left\langle 1 \quad \mathrm{c}_{2\varepsilon}\mathrm{y}D_S \quad -\mathrm{hs}_{2\varepsilon}\mathrm{y}D_S \quad 0 \right|$$

Analyser vector from Eqs. (S.7.1.1) and input Stokes vector from (E.31) yield

$$\frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(\varepsilon)\mathbf{M}_\mathrm{h} \middle| \mathbf{M}_O(\gamma)\mathbf{F}(a)\mathbf{M}_E \boldsymbol{I}_L \right\rangle}{T_S T_{rot} T_O F_{11} T_E I_L} =$$

$$= \left\langle \begin{matrix} 1 \\ \mathrm{c}_{2\varepsilon}\mathrm{y}D_S \\ -\mathrm{hs}_{2\varepsilon}\mathrm{y}D_S \\ 0 \end{matrix} \middle| \begin{matrix} i_E + D_O a\left(\mathrm{c}_{2\gamma}q_E - \mathrm{s}_{2\gamma}u_E\right) \\ \mathrm{c}_{2\gamma}D_O i_E + a q_E - \mathrm{s}_{2\gamma}\left[W_O a\left(\mathrm{s}_{2\gamma}q_E + \mathrm{c}_{2\gamma}u_E\right) + Z_O \mathrm{s}_O (1-2a) v_E\right] \\ \mathrm{s}_{2\gamma}D_O i_E - a u_E + \mathrm{c}_{2\gamma}\left[W_O a\left(\mathrm{s}_{2\gamma}q_E + \mathrm{c}_{2\gamma}u_E\right) + Z_O \mathrm{s}_O (1-2a) v_E\right] \\ Z_O \mathrm{s}_O a\left(\mathrm{s}_{2\gamma}q_E + \mathrm{c}_{2\gamma}u_E\right) + Z_O \mathrm{c}_O (1-2a) v_E \end{matrix} \right\rangle =$$

$$= \left(1 + \mathrm{y}D_S D_O \mathrm{c}_{2\gamma+\mathrm{h}2\varepsilon}\right) i_E - \mathrm{y}D_S Z_O \mathrm{s}_O \mathrm{s}_{2\gamma+\mathrm{h}2\varepsilon} v_E +$$
$$+ a\left\{D_O\left(\mathrm{c}_{2\gamma}q_E - \mathrm{s}_{2\gamma}u_E\right) + \mathrm{y}D_S\left[\left(\mathrm{c}_{2\varepsilon}q_E + \mathrm{hs}_{2\varepsilon}u_E\right) - \mathrm{s}_{2\gamma+\mathrm{h}2\varepsilon}\left(W_O\left(\mathrm{s}_{2\gamma}q_E + \mathrm{c}_{2\gamma}u_E\right) - 2Z_O \mathrm{s}_O v_E\right)\right]\right\} \tag{S.7.1.2}$$

$$\gamma = 0 \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(\varepsilon) \mathbf{M}_\mathrm{h} \middle| \mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E \boldsymbol{I}_L \right\rangle}{T_S T_{rot} T_O F_{11} T_E I_L} =$$
$$= \left(1 + \mathrm{y} D_S D_O \mathrm{c}_{\mathrm{h}2\varepsilon}\right) i_E - \mathrm{y} D_S Z_O \mathrm{s}_O \mathrm{s}_{\mathrm{h}2\varepsilon} v_E + a\left\{\left(D_O + \mathrm{y} D_S \mathrm{c}_{2\varepsilon}\right) q_E + \mathrm{y} D_S \mathrm{s}_{\mathrm{h}2\varepsilon} Z_O \left(\mathrm{c}_O u_E + 2\mathrm{s}_O v_E\right)\right\} \tag{S.7.1.3}$$

$$\gamma = \varepsilon = 0 \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(0) \mathbf{M}_\mathrm{h} \middle| \mathbf{M}_O(0) \mathbf{F}(a) \mathbf{M}_E \boldsymbol{I}_L \right\rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \left(1 + \mathrm{y} D_S D_O\right) i_E + a\left(D_O + \mathrm{y} D_S\right) q_E \tag{S.7.1.4}$$

$$\gamma = -\mathrm{h}\varepsilon \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(\varepsilon) \mathbf{M}_\mathrm{h} \middle| \mathbf{M}_O(-\mathrm{h}\varepsilon) \mathbf{F}(a) \mathbf{M}_E \boldsymbol{I}_L \right\rangle}{T_S T_{rot} T_O F_{11} T_E I_L} = \left(1 + \mathrm{y} D_S D_O\right) i_E + a\left(D_O + \mathrm{y} D_S\right)\left(\mathrm{c}_{2\gamma} q_E - \mathrm{s}_{2\gamma} u_E\right) \tag{S.7.1.5}$$

With horizontal linearly polarised emitter input Stokes vector $\boldsymbol{I}_E$

$$i_E = 1, q_E = 1, u_E = v_E = 0 \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(\varepsilon) \mathbf{M}_\mathrm{h} \mathbf{M}_O(\gamma) \mathbf{F}(a) \middle| 1 \quad 1 \quad 0 \quad 0 \right\rangle}{T_S T_{rot} T_O F_{11} T_E I_L} =$$
$$= 1 + \mathrm{y} D_S D_O \mathrm{c}_{2\gamma+\mathrm{h}2\varepsilon} + a\left\{D_O \mathrm{c}_{2\gamma} + \mathrm{y} D_S \left(\mathrm{c}_{2\varepsilon} - \mathrm{s}_{2\gamma+\mathrm{h}2\varepsilon} \mathrm{s}_{2\gamma} W_O\right)\right\} \tag{S.7.1.6}$$

With rotated, linearly polarised laser and emitter optics

$$
\begin{aligned}
& q_L = 1, u_L = 0, v_L = 0 \Rightarrow \\
& \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{R}(\varepsilon) \middle| \mathbf{M}_O(\gamma)\mathbf{F}(a)\mathbf{M}_E(\beta)\boldsymbol{I}_L(\alpha) \right\rangle}{T_S T_O F_{11} T_E I_L} = \\
& = 1 + D_E \mathrm{c}_{2\alpha-2\beta} + \mathrm{y}D_S \left\{ D_O \mathrm{c}_{2\varepsilon+2\gamma}\left(1 + D_E \mathrm{c}_{2\alpha-2\beta}\right) + Z_O \mathrm{s}_O \mathrm{s}_{2\varepsilon+2\gamma} Z_E \mathrm{s}_E \mathrm{s}_{2\alpha-2\beta} \right\} + \\
& +a\left\{\begin{array}{l}
D_O\left[\mathrm{c}_{2\gamma}\left(\mathrm{c}_{2\beta}D_E + \mathrm{c}_{2\alpha} + \mathrm{s}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right) - \mathrm{s}_{2\gamma}\left(\mathrm{s}_{2\beta}D_E + \mathrm{s}_{2\alpha} - \mathrm{c}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right)\right] + \\
+\mathrm{y}D_S\left[\begin{array}{l}
\left[\mathrm{c}_{2\varepsilon}\left(\mathrm{c}_{2\beta}D_E + \mathrm{c}_{2\alpha} + \mathrm{s}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right) + \mathrm{s}_{2\varepsilon}\left(\mathrm{s}_{2\beta}D_E + \mathrm{s}_{2\alpha} - \mathrm{c}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right)\right] - \\
-\mathrm{s}_{2\varepsilon+2\gamma}\left(\begin{array}{l}
W_O\left(\mathrm{s}_{2\gamma}\left(\mathrm{c}_{2\beta}D_E + \mathrm{c}_{2\alpha} + \mathrm{s}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right) + \mathrm{c}_{2\gamma}\left(\mathrm{s}_{2\beta}D_E + \mathrm{s}_{2\alpha} - \mathrm{c}_{2\beta}W_E\mathrm{s}_{2\alpha-2\beta}\right)\right) + \\
+2Z_O\mathrm{s}_O Z_E \mathrm{s}_E \mathrm{s}_{2\alpha-2\beta}
\end{array}\right)
\end{array}\right]
\end{array}\right\} = \\
& = \left(1 + \mathrm{c}_{2\gamma+2\varepsilon}\mathrm{y}D_S D_O\right)\left(1 + \mathrm{c}_{2\alpha-2\beta}D_E\right) + \mathrm{s}_{2\gamma+2\varepsilon}\mathrm{s}_{2\alpha-2\beta}\mathrm{y}D_S Z_O \mathrm{s}_O Z_E \mathrm{s}_E + \\
& +a\left\{\begin{array}{l}
D_O\left(\mathrm{c}_{2\alpha-2\gamma} + \mathrm{c}_{2\beta-2\gamma}D_E + \mathrm{s}_{2\beta+2\gamma}\mathrm{s}_{2\alpha-2\beta}W_E\right) + \\
+\mathrm{y}D_S\left[\begin{array}{l}
\left(\mathrm{c}_{2\alpha-2\varepsilon} + \mathrm{c}_{2\beta-2\varepsilon}D_E + \mathrm{s}_{2\beta-2\varepsilon}\mathrm{s}_{2\alpha-2\beta}W_E\right) - \\
-\mathrm{s}_{2\gamma+2\varepsilon}W_O\left(\mathrm{s}_{2\alpha+2\gamma} + \mathrm{s}_{2\beta+2\gamma}D_E - \mathrm{c}_{2\beta-2\gamma}\mathrm{s}_{2\alpha-2\beta}W_E\right) - \\
-\mathrm{s}_{2\gamma+2\varepsilon}\mathrm{s}_{2\alpha-2\beta}2Z_O\mathrm{s}_O Z_E\mathrm{s}_E
\end{array}\right]
\end{array}\right\}
\end{aligned}
\tag{S.7.1.7}
$$

$$
\begin{aligned}
& q_L = 1, u_L = 0, v_L = 0 \wedge \gamma = 0 \Rightarrow \\
& \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{R}(\varepsilon) \middle\| \mathbf{M}_O(\gamma)\mathbf{F}(a)\mathbf{M}_E(\beta)\boldsymbol{I}_L(\alpha) \right\rangle}{T_S T_O F_{11} T_E I_L} = \\
& = \left(1 + \mathrm{c}_{2\varepsilon}\mathrm{y}D_S D_O\right)\left(1 + \mathrm{c}_{2\alpha-2\beta}D_E\right) + \mathrm{s}_{2\varepsilon}\mathrm{s}_{2\alpha-2\beta}\mathrm{y}D_S Z_O \mathrm{s}_O Z_E \mathrm{s}_E + \\
& +a\left\{\begin{array}{l}
D_O\left(\mathrm{c}_{2\alpha} + \mathrm{c}_{2\beta}D_E + \mathrm{s}_{2\beta}\mathrm{s}_{2\alpha-2\beta}W_E\right) + \\
+\mathrm{y}D_S\left[\begin{array}{l}
\left(\mathrm{c}_{2\alpha-2\varepsilon} + \mathrm{c}_{2\beta-2\varepsilon}D_E + \mathrm{s}_{2\beta-2\varepsilon}\mathrm{s}_{2\alpha-2\beta}W_E\right) - \\
-\mathrm{s}_{2\varepsilon}W_O\left(\mathrm{s}_{2\alpha} + \mathrm{s}_{2\beta}D_E - \mathrm{c}_{2\beta}\mathrm{s}_{2\alpha-2\beta}W_E\right) - \\
-\mathrm{s}_{2\varepsilon}\mathrm{s}_{2\alpha-2\beta}2Z_O\mathrm{s}_O Z_E\mathrm{s}_E
\end{array}\right]
\end{array}\right\}
\end{aligned}
\tag{S.7.1.8}
$$

$$
\begin{aligned}
& q_L = 1, u_L = 0, v_L = 0, \varepsilon = -\gamma \Rightarrow \\
& \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{R}(-\gamma) \middle\| \mathbf{M}_O(\gamma)\mathbf{F}(a)\mathbf{M}_E(\beta)\boldsymbol{I}_L(\alpha) \right\rangle}{T_S T_O F_{11} T_E I_L} = \\
& = \left(1 + \mathrm{y}D_S D_O\right)\left(1 + \mathrm{c}_{2\alpha-2\beta}D_E\right) + \\
& +a\left\{D_O\left(\mathrm{c}_{2\alpha-2\gamma} + \mathrm{c}_{2\beta-2\gamma}D_E + \mathrm{s}_{2\beta+2\gamma}\mathrm{s}_{2\alpha-2\beta}W_E\right) + \mathrm{y}D_S\left(\mathrm{c}_{2\alpha+2\gamma} + \mathrm{c}_{2\beta+2\gamma}D_E + \mathrm{s}_{2\beta+2\gamma}\mathrm{s}_{2\alpha-2\beta}W_E\right)\right\}
\end{aligned}
\tag{S.7.1.9}
$$

$$
\begin{aligned}
&q_L = 1, u_L = 0, v_L = 0, \varepsilon = \gamma = 0 \Rightarrow \\
&\frac{\left\langle \mathbf{M}_S \mathbf{R}_{\mathrm{y}} \mathbf{R}(\varepsilon) \right\| \left| \mathbf{M}_O(\gamma) \mathbf{F}(a) \mathbf{M}_E(\beta) \boldsymbol{I}_L(\alpha) \right\rangle}{T_S T_O F_{11} T_E I_L} = \\
&= \left(1 + \mathrm{y} D_S D_O\right)\left(1 + \mathrm{c}_{2\alpha-2\beta} D_E\right) + a\left(D_O + \mathrm{y} D_S\right)\left(\mathrm{c}_{2\alpha} + \mathrm{c}_{2\beta} D_E + \mathrm{s}_{2\beta} \mathrm{s}_{2\alpha-2\beta} W_E\right)
\end{aligned}
\tag{S.7.1.10}
$$

S.7.2 Lidar signal with rotational error before the receiving optics

With Eq. (D.7) for the analyser part and (E.26) for the general input vector we get

$$
\begin{aligned}
&\frac{\left\langle \mathbf{A}_S(\mathrm{y},\gamma) \right|}{T_S T_O} \frac{\left| \boldsymbol{I}_{in,\varepsilon}(\varepsilon,\mathrm{h},a) \right\rangle}{T_{rot} F_{11} T_E I_L} = \frac{\left\langle \mathbf{M}_S \mathbf{R}_{\mathrm{y}} \mathbf{M}_O(\gamma) \right|}{T_S T_O} \frac{\left| \mathbf{R}(\varepsilon) \mathbf{M}_{\mathrm{h}} \mathbf{F}(a) \boldsymbol{I}_E \right\rangle}{T_{rot} F_{11} T_E I_L} = \\
&= \left\langle \begin{matrix} 1 + \mathrm{y}\mathrm{c}_{2\gamma} D_O D_S \\ \mathrm{c}_{2\gamma} D_O + \mathrm{y} D_S \left(1 - \mathrm{s}_{2\gamma}^2 W_O\right) \\ \mathrm{s}_{2\gamma}\left(D_O + \mathrm{y}\mathrm{c}_{2\gamma} D_S W_O\right) \\ -\mathrm{y}\mathrm{s}_{2\gamma} D_S Z_O \mathrm{s}_O \end{matrix} \right\| \left| \begin{matrix} i_E \\ a\left(q_E \mathrm{c}_{2\varepsilon} + \mathrm{h} u_E \mathrm{s}_{2\varepsilon}\right) \\ a\left(q_E \mathrm{s}_{2\varepsilon} - \mathrm{h} u_E \mathrm{c}_{2\varepsilon}\right) \\ (1-2a)\mathrm{h} v_E \end{matrix} \right\rangle = \\
&= \left(1 + \mathrm{y}\mathrm{c}_{2\gamma} D_O D_S\right) i_E - \mathrm{y}\mathrm{s}_{2\gamma} D_S Z_O \mathrm{s}_O \mathrm{h} v_E + \\
&+a \left\{ \begin{matrix} D_O\left[\mathrm{c}_{2\gamma-2\varepsilon} q_E - \mathrm{s}_{2\gamma-2\varepsilon} \mathrm{h} u_E\right] - \mathrm{y} D_S W_O \mathrm{s}_{2\gamma}\left[\mathrm{s}_{2\gamma-2\varepsilon} q_E + \mathrm{c}_{2\gamma-2\varepsilon} \mathrm{h} u_E\right] + \\ +\mathrm{y} D_S\left[q_E \mathrm{c}_{2\varepsilon} + \mathrm{h} u_E \mathrm{s}_{2\varepsilon} + 2\mathrm{s}_{2\gamma} Z_O \mathrm{s}_O \mathrm{h} v_E\right] \end{matrix} \right\}
\end{aligned}
\tag{S.7.2.1}
$$

Comparison with Eq. (69):

$$
\begin{aligned}
&\varepsilon = 0 \Rightarrow \\
&\frac{\left\langle \mathbf{M}_S \mathbf{R}_{\mathrm{y}} \mathbf{M}_O(\gamma) \right|}{T_S T_O} \frac{\left| \mathbf{R}(\varepsilon) \mathbf{M}_{\mathrm{h}} \mathbf{F}(a) \boldsymbol{I}_E \right\rangle}{T_{rot} F_{11} T_E I_L} = \\
&= \left(1 + \mathrm{y}\mathrm{c}_{2\gamma} D_O D_S\right) i_E - \mathrm{y}\mathrm{s}_{2\gamma} D_S Z_O \mathrm{s}_O \mathrm{h} v_E + \\
&+a \left\{ \begin{matrix} D_O\left[\mathrm{c}_{2\gamma} q_E - \mathrm{s}_{2\gamma} \mathrm{h} u_E\right] - \mathrm{y} D_S W_O \mathrm{s}_{2\gamma}\left[\mathrm{s}_{2\gamma} q_E + \mathrm{c}_{2\gamma} \mathrm{h} u_E\right] + \\ +\mathrm{y} D_S\left[q_E + 2\mathrm{s}_{2\gamma} Z_O \mathrm{s}_O \mathrm{h} v_E\right] \end{matrix} \right\}
\end{aligned}
\tag{S.7.2.2}
$$

$$
\begin{aligned}
&\gamma = 0 \Rightarrow \\
&\frac{\left\langle \mathbf{M}_S \mathbf{R}_{\mathrm{y}} \mathbf{M}_O(\gamma) \right|}{T_S T_O} \frac{\left| \mathbf{R}(\varepsilon) \mathbf{M}_{\mathrm{h}} \mathbf{F}(a) \boldsymbol{I}_E \right\rangle}{T_{rot} F_{11} T_E I_L} = \left(1 + \mathrm{y} D_O D_S\right) i_E + a\left(D_O + \mathrm{y} D_S\right)\left[\mathrm{c}_{2\varepsilon} q_E + \mathrm{s}_{2\varepsilon} \mathrm{h} u_E\right]
\end{aligned}
\tag{S.7.2.3}
$$

S.7.3 Lidar signal with rotational error behind the emitter optics

We get the equation for this case directly from the previous one considering that moving the matrices for the rotational error from before the receiving optics to behind the emitter optics just changes the sign of the angle ε using Eq. (S.6.2)

$$\frac{\langle \mathbf{A}_S(\mathrm{y},\gamma,a)|}{T_O T_S F_{11}} \frac{|\boldsymbol{I}_{in,\varepsilon}(\varepsilon,\mathrm{h})\rangle}{T_{rot} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{M}_O(\gamma) \mathbf{F}(a)|}{T_O T_S F_{11}} \frac{|\mathbf{R}(\varepsilon) \mathbf{M}_\mathrm{h} \boldsymbol{I}_E\rangle}{T_{rot} T_E I_L} = \frac{\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{M}_O(\gamma) \mathbf{R}(-\varepsilon) \mathbf{M}_\mathrm{h} \| \mathbf{F}(a) \boldsymbol{I}_E\rangle}{T_O T_S F_{11} T_{rot} T_E I_L} = \frac{\langle \mathbf{A}_S(\mathrm{y},\gamma)|}{T_S T_O} \frac{|\boldsymbol{I}_{in,\varepsilon}(-\varepsilon,\mathrm{h},a)\rangle}{T_{rot} F_{11} T_E I_L} \tag{S.7.3.1}$$

S.8 Attenuated backscatter coefficient

Attenuated backscatter coefficient F_{11} derived from the transmitted signal I_T

$$\eta_T T_T T_O F_{11} T_E I_L = \frac{I_T}{G_T + aH_T} = \frac{I_T}{G_T + \frac{\delta^* G_T - G_R}{H_R - \delta^* H_T} H_T} = \frac{\left(H_R - \delta^* H_T\right) I_T}{\left(H_R - \delta^* H_T\right) G_T + \left(\delta^* G_T - G_R\right) H_T} = \frac{\left(H_R - \delta^* H_T\right) I_T}{H_R G_T - H_T G_R} = \frac{\left(H_R - \frac{1}{\eta}\frac{I_R}{I_T} H_T\right) I_T}{H_R G_T - H_T G_R} = \frac{H_R I_T - \frac{1}{\eta} I_R H_T}{H_R G_T - H_T G_R} \tag{S.8.1}$$

With $\eta = \eta_R T_R / \eta_T T_T$ we get the attenuated backscatter coefficient

$$F_{11} = \frac{1}{\eta_T T_T T_O I_L} \frac{H_R I_T - \frac{\eta_T T_T}{\eta_R T_R} H_T I_R}{H_R G_T - H_T G_R} = \frac{1}{T_O I_L} \frac{H_R \frac{I_T}{\eta_T T_T} - H_T \frac{I_R}{\eta_R T_R}}{H_R G_T - H_T G_R} \tag{S.8.2}$$

S.9 Rayleigh calibration

Calibration in ranges with presumably known aerosol depolarisation:

With some lidar systems the calibration factor is determined in a measurement range with known linear volume depolarisation ratio δ, for example in clean air δ^m. Assuming, for the sake of simplicity, an ideal PBS (see Eq. (28)), we get with Eq. (26) for the calibration factor in clean air η^m

$$\delta^*(0°) = \frac{1}{\eta}\frac{I_R}{I_T}(0°) \Rightarrow \eta^m = \frac{1}{\delta^m}\frac{I_R}{I_T}(0°) \tag{S.9.1}$$

Assuming further that the errors in I_R and I_T are independent, which could be the case if the background subtraction or nonlinearities in analog signal detection are the main error sources for them, we get as a first estimate for the relative error of the calibration factor

$$\frac{\Delta\eta^m}{\eta^m} = \frac{\Delta\delta^m}{\delta^m} + \frac{\Delta I_R}{I_R} + \frac{\Delta I_T}{I_T} \tag{S.9.2}$$

This error can easily become very large. The linear depolarisation ratio measured in a volume of air molecules δ^m depends on the width of the interference filters, as they transmit or reject some rotational Raman lines, and on the atmospheric temperature Behrendt and Nakamura (2002). At 355 nm δ^m can range from about 0.004 to 0.015. Errors in the order of some 10% in δ^m are already possible in case the wavelength dependence of the transmission of the IFF or its tilt angle in the optical setup are not known accurately. Furthermore, a small contamination of the assumed clean air with strongly depolarising aerosol as Saharan dust or ice particles from subvisible cirrus can change the volume linear depolarisation ratio dramatically. Assuming a small backscatter ratio of 1.01 due to particles with δ^p = 0.3 and with δ^m = 0.004, we get Biele et al. (2000) a real δ = 0.01* δ^p + δ^m = 0.007, which would cause a relative error in the calibration factor η^m of (0.007-0.004)/0.004 = +75%. Better than this "clean" air calibration would even be to use a calibration in a cirrus cloud with δ^p between let's say 0.3 and 0.5, with a resulting calibration factor error of "only" +-0.1/0.4 = +-25%.

Summary of this chapter: Depolarisation calibration with presumably known atmospheric depolarisation can cause very large calibration errors.

S.10 Some Müller matrices

S.10.1 Depolariser

A diagonal depolariser $\mathbf{M}_{DD}$ (Chipman, 2009b)

$$\mathbf{M}_{DD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \tag{S.10.1.1}$$

partially depolarises the incident light depending on its state of polarisation. For atmospheric depolarisation by randomly oriented, nonspherical particles with rotation and reflection symmetry it can be shown that $b = -a$ and $c = (1 - 2a)$ (van de Hulst, 1981; Mishchenko and Hovenier, 1995; Mishchenko et al., 2002) (see also Sect. 2.1), which results in the backscattering matrix

$$\mathbf{F} = \begin{pmatrix} F_{11} & 0 & 0 & 0 \\ 0 & F_{22} & 0 & 0 \\ 0 & 0 & -F_{22} & 0 \\ 0 & 0 & 0 & F_{44} \end{pmatrix} = F_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \tag{S.10.1.2}$$

S.10.2 Rotation matrix for various rotation angles

$$\mathbf{R}(\phi)=\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{s}_{2\phi}&0\\0&\mathrm{s}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix},\quad \mathbf{R}(\mathrm{x}\phi)=\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{xs}_{2\phi}&0\\0&\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}$$
$$\mathbf{R}(\mathrm{x}45^\circ)=\begin{pmatrix}1&0&0&0\\0&0&-\mathrm{x}&0\\0&\mathrm{x}&0&0\\0&0&0&1\end{pmatrix}\tag{S.10.2.1}$$

$$\mathbf{R}(\mathrm{x},\varepsilon)=\mathbf{R}(\mathrm{x}45^\circ+\varepsilon)=\mathbf{R}(\mathrm{x}45^\circ)\mathbf{R}(\varepsilon)=\mathbf{R}(\varepsilon)\mathbf{R}(\mathrm{x}45^\circ)=\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&-\mathrm{xc}_{2\varepsilon}&0\\0&\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}\tag{S.10.2.2}$$

See Supp. S.10.15. regarding the half-wave plate rotation.

$$\mathbf{R}(90^\circ)=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},\quad \mathbf{R}_{\mathrm{y}}=\mathbf{R}(\mathrm{y})\equiv\begin{pmatrix}1&0&0&0\\0&\mathrm{y}&0&0\\0&0&\mathrm{y}&0\\0&0&0&1\end{pmatrix}\Rightarrow\begin{matrix}\mathbf{R}(\mathrm{y}=-1)=\mathbf{R}(90^\circ)\\\mathbf{R}(\mathrm{y}=+1)=\mathbf{R}(0^\circ)\end{matrix}\tag{S.10.2.3}$$

S.10.3 Retarding linear diattenuator

$$\mathbf{M}_O=\mathbf{M}_D\mathbf{M}_{ret}=\mathbf{M}_{ret}\mathbf{M}_D=$$
$$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\mathrm{c}_O&\mathrm{s}_O\\0&0&-\mathrm{s}_O&\mathrm{c}_O\end{pmatrix}T_O\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&Z_O&0\\0&0&0&Z_O\end{pmatrix}=T_O\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&Z_O\mathrm{c}_O&Z_O\mathrm{s}_O\\0&0&-Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}\tag{S.10.3.1}$$

$$D_O=\frac{T_O^p-T_O^s}{T_O^p+T_O^s},\quad Z_O=\sqrt{1-D_O^2},\quad W_O=1-Z_O\mathrm{c}_O$$
$$\mathrm{c}_O=\cos\Delta_O=\cos\left(\varphi_O^p-\varphi_O^s\right),\quad \mathrm{s}_O=\sin\Delta_O\tag{S.10.3.2}$$

$$-1\le D_O\le+1\Rightarrow\quad 0\le Z_O\le1,\quad 0\le W_O\le2\tag{S.10.3.3}$$

S.10.4 Rotated, retarding linear diattenuator

$$\mathbf{M}_O(\mathrm{x}\phi)=\mathbf{R}(\mathrm{x}\phi)\mathbf{M}_O\mathbf{R}(-\mathrm{x}\phi)=$$

$$=T_O\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{xs}_{2\phi}&0\\0&\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&Z_O\mathrm{c}_O&Z_O\mathrm{s}_O\\0&0&-Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&\mathrm{xs}_{2\phi}&0\\0&-\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}= \tag{S.10.4.1}$$

$$=T_O\begin{pmatrix}1&\mathrm{c}_{2\phi}D_O&\mathrm{xs}_{2\phi}D_O&0\\\mathrm{c}_{2\phi}D_O&1-\mathrm{s}_{2\phi}^2W_O&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&-\mathrm{xs}_{2\phi}Z_Os_O\\\mathrm{xs}_{2\phi}D_O&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&1-\mathrm{c}_{2\phi}^2W_O&\mathrm{c}_{2\phi}Z_Os_O\\0&\mathrm{xs}_{2\phi}Z_Os_O&-\mathrm{c}_{2\phi}Z_Os_O&Z_Oc_O\end{pmatrix}$$

S.10.5 Rotated, retarding linear diattenuator mirror

$$\mathbf{M}_{MO}(\mathrm{x}\phi)=\mathbf{R}(\mathrm{x}\phi)\mathbf{M}_{MO}\mathbf{R}(\mathrm{x}\phi)=$$

$$=T_O\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{xs}_{2\phi}&0\\0&\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&-Z_O\mathrm{c}_O&-Z_O\mathrm{s}_O\\0&0&Z_O\mathrm{s}_O&-Z_O\mathrm{c}_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{xs}_{2\phi}&0\\0&\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}=$$

$$=T_O\begin{pmatrix}1&\mathrm{c}_{2\phi}D_O&-\mathrm{xs}_{2\phi}D_O&0\\\mathrm{c}_{2\phi}D_O&1-\mathrm{s}_{2\phi}^2W_O&-\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&\mathrm{xs}_{2\phi}Z_O\mathrm{s}_O\\\mathrm{xs}_{2\phi}D_O&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&-\left(1-\mathrm{c}_{2\phi}^2W_O\right)&-\mathrm{c}_{2\phi}Z_Os_O\\0&\mathrm{xs}_{2\phi}Z_Os_O&\mathrm{c}_{2\phi}Z_Os_O&-Z_Oc_O\end{pmatrix}= \tag{S.10.5.1}$$

$$=T_O\begin{pmatrix}1&\mathrm{c}_{2\phi}D_O&\mathrm{xs}_{2\phi}D_O&0\\\mathrm{c}_{2\phi}D_O&1-\mathrm{s}_{2\phi}^2W_O&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&-\mathrm{xs}_{2\phi}Z_Os_O\\\mathrm{xs}_{2\phi}D_O&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}W_O&1-\mathrm{c}_{2\phi}^2W_O&\mathrm{c}_{2\phi}Z_Os_O\\0&\mathrm{xs}_{2\phi}Z_Os_O&-\mathrm{c}_{2\phi}Z_Os_O&Z_Oc_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

S.10.6 ±45° rotated retarding linear diattenuator including error ε

$$\mathbf{M}_O\left(\mathrm{x}45°+\varepsilon\right)=\mathbf{R}\left(\mathrm{x}45°+\varepsilon\right)\mathbf{M}_O\mathbf{R}\left(-\mathrm{x}45°-\varepsilon\right)=$$

$$=T_O\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&-\mathrm{xc}_{2\varepsilon}&0\\0&\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&D_O&0&0\\D_O&1&0&0\\0&0&Z_O\mathrm{c}_O&Z_O\mathrm{s}_O\\0&0&-Z_O\mathrm{s}_O&Z_O\mathrm{c}_O\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\0&-\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}= \tag{S.10.6.1}$$

$$=T_O\begin{pmatrix}1&-\mathrm{xs}_{2\varepsilon}D_O&\mathrm{xc}_{2\varepsilon}D_O&0\\-\mathrm{xs}_{2\varepsilon}D_O&1-\mathrm{c}_{2\varepsilon}^2W_O&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&-\mathrm{xc}_{2\varepsilon}Z_Os_O\\\mathrm{xc}_{2\varepsilon}D_O&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_O&1-\mathrm{s}_{2\varepsilon}^2W_O&-\mathrm{xs}_{2\varepsilon}Z_Os_O\\0&\mathrm{xc}_{2\varepsilon}Z_Os_O&\mathrm{xs}_{2\varepsilon}Z_Os_O&Z_Oc_O\end{pmatrix}$$

S.10.7 ±45° rotated retarding linear diattenuator

$$\mathbf{M}_O\left(\mathrm{x}45°\right)=T_O\begin{pmatrix}1&0&\mathrm{x}D_O&0\\0&Z_Oc_O&0&-\mathrm{x}Z_Os_O\\\mathrm{x}D_O&0&1&0\\0&\mathrm{x}Z_Os_O&0&Z_Oc_O\end{pmatrix} \tag{S.10.7.1}$$

S.10.8 Rotated, ideal linear polariser and analyser

Note: without absorption $T_P = 0.5$.

$$D_P=1, Z_P=0, W_P=1\Rightarrow$$

$$\frac{\mathbf{M}_P\left(0°\right)}{T_P}=\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\left|\begin{matrix}1\\1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right| \tag{S.10.8.1}$$

$$\frac{\mathbf{M}_P\left(\mathrm{x}\phi\right)}{T_P}=\frac{\mathbf{R}\left(\mathrm{x}\phi\right)\mathbf{M}_P\mathbf{R}\left(-\mathrm{x}\phi\right)}{T_P}=$$

$$=\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{xs}_{2\phi}&0\\0&\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}\left|\begin{matrix}1\\1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right|\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&\mathrm{xs}_{2\phi}&0\\0&-\mathrm{xs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}= \tag{S.10.8.2}$$

$$=\begin{pmatrix}1&\mathrm{c}_{2\phi}&\mathrm{xs}_{2\phi}&0\\\mathrm{c}_{2\phi}&\mathrm{c}_{2\phi}^2&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}&0\\\mathrm{xs}_{2\phi}&\mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}&\mathrm{s}_{2\phi}^2&0\\0&0&0&0\end{pmatrix}=\left|\begin{matrix}1\\\mathrm{c}_{2\phi}\\\mathrm{xs}_{2\phi}\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\\mathrm{c}_{2\phi}\\\mathrm{xs}_{2\phi}\\0\end{matrix}\right|$$

1 $$\frac{\mathbf{M}_P\left(\mathrm{x}45^\circ\right)}{T_P}=\begin{pmatrix}1&0&\mathrm{x}&0\\0&0&0&0\\\mathrm{x}&0&1&0\\0&0&0&0\end{pmatrix}=\left|\begin{matrix}1\\0\\\mathrm{x}\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\0\\\mathrm{x}\\0\end{matrix}\right| \tag{S.10.8.3}$$

2 $$\frac{\mathbf{M}_P\left(90^\circ\right)}{T_P}=\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\left|\begin{matrix}1\\-1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-1\\0\\0\end{matrix}\right| \tag{S.10.8.4}$$

$$D_P=1,Z_P=0,W_P=1\Rightarrow$$

3 $$\frac{\mathbf{M}_P\left(45^\circ-\mathrm{x}45^\circ\right)}{T_P}=\begin{pmatrix}1&\mathrm{x}&0&0\\\mathrm{x}&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\left|\begin{matrix}1\\\mathrm{x}\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\\mathrm{x}\\0\\0\end{matrix}\right| \tag{S.10.8.5}$$

$$D_P=1,Z_P=0,W_P=1\Rightarrow$$

$$\mathbf{M}_P\left(\mathrm{x}45^\circ+\varepsilon\right)=\mathbf{R}\left(+\varepsilon\right)\mathbf{M}_P\left(\mathrm{x}45^\circ\right)\mathbf{R}\left(-\varepsilon\right)=$$

4 $$=T_P\begin{pmatrix}1&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\-\mathrm{xs}_{2\varepsilon}&\mathrm{s}_{2\varepsilon}^2&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}&0\\\mathrm{xc}_{2\varepsilon}&-\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}&\mathrm{c}_{2\varepsilon}^2&0\\0&0&0&0\end{pmatrix}=T_P\left|\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right| \tag{S.10.8.6}$$

$$D_P=1,Z_P=0,W_P=1\Rightarrow$$

5 $$\frac{\mathbf{M}_P\left(\mathrm{x}45^\circ+\varepsilon\right)\boldsymbol{I}_{in}}{T_PI_{in}}=\left|\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right|\left|\begin{matrix}i_{in}\\q_{in}\\u_{in}\\v_{in}\end{matrix}\right\rangle=\left|\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right\rangle\left[i_{in}-\mathrm{x}\left(\mathrm{s}_{2\varepsilon}q_{in}-\mathrm{c}_{2\varepsilon}u_{in}\right)\right] \tag{S.10.8.7}$$

$$\frac{\mathbf{M}_P\left(\mathrm{x}45^\circ+\varepsilon\right)\mathbf{F}\left(a\right)\boldsymbol{I}_{in}}{T_PF_{11}I_{in}}=$$

6 $$=\left|\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right|\begin{pmatrix}1&0&0&0\\0&a&0&0\\0&0&-a&0\\0&0&0&1-2a\end{pmatrix}\left|\begin{matrix}i_{in}\\q_{in}\\u_{in}\\v_{in}\end{matrix}\right\rangle=\left|\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right\rangle\left[i_{in}-\mathrm{x}a\left(\mathrm{s}_{2\varepsilon}q_{in}+\mathrm{c}_{2\varepsilon}u_{in}\right)\right] \tag{S.10.8.8}$$

$$\frac{\mathbf{F}(a)\mathbf{M}_P(\mathrm{x}45°+\varepsilon)\boldsymbol{I}_{in}}{T_P F_{11} I_{in}} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 1-2a \end{pmatrix} \left| \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right\| \left. \begin{matrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{matrix} \right\rangle = \left| \begin{matrix} 1 \\ -\mathrm{xas}_{2\varepsilon} \\ -\mathrm{xac}_{2\varepsilon} \\ 0 \end{matrix} \right\rangle \left[i_{in} - \mathrm{x}\left(\mathrm{s}_{2\varepsilon} q_{in} - \mathrm{c}_{2\varepsilon} u_{in}\right) \right] \quad \text{(S.10.8.9)}$$

For the 0° and 90° measurements with a perfect polariser $\mathbf{M}_P$ we get from Eq. (S.10.8.1) and Eq. (S.12.2)

$$\begin{aligned}
&for\ \ D_P = 1, Z_P = 0, W_P = 1 \\
&\mathbf{M}_P(-\mathrm{x}45° + 45° + \varepsilon)\boldsymbol{I}_{in} = \\
&= T_P I_{in} \begin{pmatrix} 1 & \mathrm{xc}_{2\varepsilon} & \mathrm{xs}_{2\varepsilon} & 0 \\ \mathrm{xc}_{2\varepsilon} & \mathrm{c}_{2\varepsilon}^2 & \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} & 0 \\ \mathrm{xs}_{2\varepsilon} & \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} & \mathrm{s}_{2\varepsilon}^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left| \begin{matrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{matrix} \right\rangle = T_P I_{in} \left| \begin{matrix} i_{in} + \mathrm{x}\left(\mathrm{c}_{2\varepsilon} q_{in} - \mathrm{s}_{2\varepsilon} u_{in}\right) \\ \mathrm{xc}_{2\varepsilon} i_{in} + \mathrm{c}_{2\varepsilon}^2 q_{in} + \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} u_{in} \\ \mathrm{xs}_{2\varepsilon} i_{in} + \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} q_{in} + \mathrm{s}_{2\varepsilon}^2 u_{in} \\ 0 \end{matrix} \right\rangle = \\
&= T_P I_{in} \left| \begin{matrix} i_{in} + \mathrm{x}\left(\mathrm{c}_{2\varepsilon} q_{in} - \mathrm{s}_{2\varepsilon} u_{in}\right) \\ \mathrm{xc}_{2\varepsilon} i_{in} + \mathrm{c}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon} q_{in} - \mathrm{s}_{2\varepsilon} u_{in}\right) \\ \mathrm{xs}_{2\varepsilon} i_{in} + \mathrm{s}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon} q_{in} - \mathrm{s}_{2\varepsilon} u_{in}\right) \\ 0 \end{matrix} \right\rangle = T_P I_{in} \left[i_{in} + \mathrm{x}\left(\mathrm{c}_{2\varepsilon} q_{in} - \mathrm{s}_{2\varepsilon} u_{in}\right) \right] \left| \begin{matrix} 1 \\ \mathrm{xc}_{2\varepsilon} \\ \mathrm{xs}_{2\varepsilon} \\ 0 \end{matrix} \right\rangle
\end{aligned} \quad \text{(S.10.8.10)}$$

S.10.9 Two rotated retarding linear diattenuators

$$\begin{aligned}
&\mathbf{M}_A(\phi)\mathbf{M}_O(\gamma) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A\mathbf{R}(-\phi)\mathbf{R}(\gamma)\mathbf{M}_O \qquad \mathbf{R}(-\gamma) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma-\phi)\mathbf{M}_O \qquad \mathbf{R}(-\gamma) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma-\phi)\mathbf{M}_O \qquad \mathbf{R}(-\gamma)\mathbf{R}(\phi)\mathbf{R}(-\phi) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \quad \mathbf{R}(\gamma-\phi)\mathbf{M}_O \qquad \mathbf{R}(\phi-\gamma)\mathbf{R}(-\phi) = \\
&= \mathbf{R}(\phi)\mathbf{M}_A \qquad \mathbf{M}_O(\gamma-\phi)\mathbf{R}(-\phi)
\end{aligned} \quad \text{(S.10.9.1)}$$

$$\frac{\mathbf{M}_A(\phi)\mathbf{M}_O(\gamma)}{T_A T_O} =$$

$$= \begin{pmatrix} 1 & \mathrm{c}_{2\phi} D_A & \mathrm{s}_{2\phi} D_A & 0 \\ \mathrm{c}_{2\phi} D_A & 1-\mathrm{s}_{2\phi}^2 W_A & \mathrm{s}_{2\phi}\mathrm{c}_{2\phi} W_A & -\mathrm{s}_{2\phi} Z_A \mathrm{s}_A \\ \mathrm{s}_{2\phi} D_A & \mathrm{s}_{2\phi}\mathrm{c}_{2\phi} W_A & 1-\mathrm{c}_{2\phi}^2 W_A & \mathrm{c}_{2\phi} Z_A \mathrm{s}_A \\ 0 & \mathrm{s}_{2\phi} Z_A \mathrm{s}_A & -\mathrm{c}_{2\phi} Z_A \mathrm{s}_A & Z_A \mathrm{c}_A \end{pmatrix} \begin{pmatrix} 1 & \mathrm{c}_{2\gamma} D_O & \mathrm{s}_{2\gamma} D_O & 0 \\ \mathrm{c}_{2\gamma} D_O & 1-\mathrm{s}_{2\gamma}^2 W_O & \mathrm{s}_{2\gamma}\mathrm{c}_{2\gamma} W_O & -\mathrm{s}_{2\gamma} Z_O \mathrm{s}_O \\ \mathrm{s}_{2\gamma} D_O & \mathrm{s}_{2\gamma}\mathrm{c}_{2\gamma} W_O & 1-\mathrm{c}_{2\gamma}^2 W_O & \mathrm{c}_{2\gamma} Z_O \mathrm{s}_O \\ 0 & \mathrm{s}_{2\gamma} Z_O \mathrm{s}_O & -\mathrm{c}_{2\gamma} Z_O \mathrm{s}_O & Z_O \mathrm{c}_O \end{pmatrix}$$

(S.10.9.2)

The first row vector of Eq. (S.10.9.2)

$$\frac{\left\langle \mathbf{M}_A(\phi)\mathbf{M}_O(\gamma)\right|}{T_S T_O} = \left\langle \begin{matrix} 1+\mathrm{c}_{2\gamma-2\phi}D_A D_O \\ \mathrm{c}_{2\gamma}D_O+\left(1-\mathrm{s}_{2\gamma}^2 W_O\right)\mathrm{c}_{2\phi}D_A+\mathrm{s}_{2\gamma}\mathrm{c}_{2\gamma}W_O\mathrm{s}_{2\phi}D_A \\ \mathrm{s}_{2\gamma}\left(D_O+\mathrm{c}_{2\gamma}W_O\mathrm{c}_{2\phi}D_A\right)+\left(1-\mathrm{c}_{2\gamma}^2 W_O\right)\mathrm{s}_{2\phi}D_A \\ -\mathrm{s}_{2\gamma}Z_O\mathrm{s}_O\mathrm{c}_{2\phi}D_A+\mathrm{c}_{2\gamma}Z_O\mathrm{s}_O\mathrm{s}_{2\phi}D_A \end{matrix}\right| = \left\langle \begin{matrix} 1+\mathrm{c}_{2\gamma-2\phi}D_A D_O \\ \mathrm{c}_{2\gamma}D_O+\left(\mathrm{c}_{2\phi}-\mathrm{s}_{2\gamma}\mathrm{s}_{2\gamma-2\phi}W_O\right)D_A \\ \mathrm{s}_{2\gamma}D_O+\left(\mathrm{s}_{2\phi}+\mathrm{c}_{2\gamma}\mathrm{s}_{2\gamma-2\phi}W_O\right)D_A \\ -\mathrm{s}_{2\gamma-2\phi}Z_O\mathrm{s}_O D_A \end{matrix}\right| \quad \text{(S.10.9.3)}$$

S.10.10 Cleaned analyser (polarising beam-splitter with additional polarising sheet filters)

The intensity transmission of analysers is proportional to a certain state of polarisation before the analyser, with arbitrary state of polarisation behind, while the output of polarisers is a certain state of polarisation regardless which state of polarisation exists before the polariser (Lu and Chipman, 1996). Here we use a polarising sheet filter, which is a depolarising analyser and a depolarising polariser at the same time, to get rid of the cross talk of the polarising beam-splitter. The combined matrix of a polarising sheet filter $\mathbf{M}_A$ behind the polarising beam-splitter $\mathbf{M}_S$ is again the matrix of a retarding linear diattenuator, which we call a cleaned polarising beam-splitter. If $\mathbf{M}_A$ is rotated (misaligned) by ϕ we get from Eq. (S.10.9.3)

$$\gamma = 0 \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_A(\phi)\mathbf{M}_S(0)\right|}{T_A T_S} = \left\langle 1+\mathrm{c}_{2\phi}D_A D_S \quad D_S+\mathrm{c}_{2\phi}D_A \quad \mathrm{s}_{2\phi}D_A Z_S\mathrm{c}_S \quad \mathrm{s}_{2\phi}D_A Z_S\mathrm{s}_S\right| \quad \text{(S.10.10.1)}$$

$$\gamma = 0, \phi = 0 \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_A(0)\mathbf{M}_S(0)\right|}{T_A T_S} = \left\langle 1+D_A D_S \quad D_S+D_A \quad 0 \quad 0\right| \quad \text{(S.10.10.2)}$$

$$\gamma = 0, \phi = 90° \Rightarrow$$
$$\frac{\left\langle \mathbf{M}_A(90°)\mathbf{M}_S(0)\right|}{T_A T_S} = \left\langle 1-D_A D_S \quad D_S-D_A \quad 0 \quad 0\right| \quad \text{(S.10.10.3)}$$

Typically the manufacturers' terminology is as Eq. (S.10.10.4) and their specifications for polarising sheet filters are the transmission of two crossed filters Eq. (S.10.10.6) and that of two parallel filters Eq. (S.10.10.7) of the same type.

$$T_A^p = k_1 \text{ and } T_A^s = k_2 \quad \Rightarrow \tag{S.10.10.4}$$

$$D_A = \frac{k_1 - k_2}{k_1 + k_2},\ Z_A = \frac{2\sqrt{k_1 k_2}}{k_1 + k_2},\ T_A = \frac{k_1 + k_2}{2} \tag{S.10.10.5}$$

$$T_{cross} = H_{90} = k_1 k_2 = T_A \sqrt{1 - D_A^{\ 2}} \tag{S.10.10.6}$$

$$T_{parallel} = H_0 = 0.5\left(k_1^{\ 2} + k_2^{\ 2}\right) = T_A \sqrt{1 + D_A^{\ 2}} \tag{S.10.10.7}$$

For the extinction ratio ρ (see Bennett (2009a), Sect. 12.4) and its inverse, i.e. the contrast ratio or transmission rato, different definitions, as in Eq. (S.10.10.9), can be found in manufacturers' descriptions, which is sometimes confusing. However, usually $k_2 \ll k_1$, and the given extinction ratios are then to be understood as "on the order of", irrespective of the used formula.

$$\rho = \frac{k_2}{k_1} \tag{S.10.10.8}$$

$$k_2 \ll k_1 \Rightarrow$$
$$\frac{T_{cross}}{T_{parallel}} = \frac{H_{90}}{H_0} = \frac{k_1 k_2}{0.5\left(k_1^{\ 2} + k_2^{\ 2}\right)} \approx 2\rho \tag{S.10.10.9}$$

$$D_T^{\#} = \frac{D_T + D_A}{1 + D_T D_A} = \frac{T_T^p k_1 - T_T^s k_2}{T_T^p k_1 + T_T^s k_2}, \qquad D_R^{\#} = \frac{D_R - D_A}{1 - D_R D_A} = \frac{T_R^p k_2 - T_R^s k_1}{T_R^p k_2 + T_R^s k_1}$$
$$T_T^{\#} = T_T T_A \left(1 + D_T D_A\right) = 0.5\left(T_T^p k_1 + T_T^s k_2\right), \quad T_R^{\#} = T_R T_A \left(1 - D_R D_A\right) = 0.5\left(T_R^p k_2 + T_R^s k_1\right)$$
$$Z_T^{\#} = \frac{Z_T Z_A}{1 + D_T D_A} = \frac{2\sqrt{T_T^p k_1 T_T^s k_2}}{T_T^p k_1 + T_T^s k_2}, \qquad Z_R^{\#} = \frac{Z_R Z_A}{1 - D_R D_A} = \frac{2\sqrt{T_R^s k_1 T_R^p k_2}}{T_R^p k_2 + T_R^s k_1}$$
(S.10.10.10)

For an ideal (cleaned) analyser $\mathbf{M}_A$ with total extinction ($k_2 = 0$) we get from Eqs. (S.10.10.3) and (S.10.10.10)

$$\text{with } k_2 = 0, D_A = 1, Z_A = 0 \Rightarrow$$
$$T_T^{\#} = 0.5 T_T^p k_1,\ T_R^{\#} = 0.5 T_R^s k_1,\ D_T^{\#} = +1,\ D_R^{\#} = -1,\ Z_S^{\#} = 0 \tag{S.10.10.11}$$

$$\eta = \frac{\eta_R T_R^{\#}}{\eta_T T_T^{\#}} = \frac{\eta_R T_R^s}{\eta_T T_T^p} \tag{S.10.10.12}$$

General:

$$\langle \mathbf{M}_{21}| \equiv \langle \mathbf{M}_2(0°)\mathbf{M}_1(0°)| = T_2T_1\langle 1+D_2D_1 \quad D_2+D_1 \quad 0 \quad 0| = T_{21}\langle 1 \quad D_{21} \quad 0 \quad 0|$$

$$D_1 = \frac{T_1^p - T_1^s}{T_1^p + T_1^s}$$

$$T_1 = 0.5\left(T_1^p + T_1^s\right)$$

$$D_{21} = \frac{D_2+D_1}{1+D_2D_1} = \frac{T_2^pT_1^p - T_2^sT_1^s}{T_2^pT_1^p + T_2^sT_1^s}, \tag{S.10.10.13}$$

$$T_{21} = T_2T_1\left(1+D_2D_1\right) = 0.5\left(T_2^pT_1^p + T_2^sT_1^s\right)$$

$$Z_{21} = \frac{Z_2Z_1}{1+D_2D_1} = \frac{2\sqrt{T_2^pT_1^pT_2^sT_1^s}}{T_2^pT_1^p + T_2^sT_1^s}$$

$$1-D_1 = \frac{T_1^s}{T_1}, 1+D_1 = \frac{T_1^p}{T_1} \tag{S.10.10.14}$$

$$D_{SyO} = \frac{D_O + yD_S}{1+yD_SD_O} = \frac{(1+y)\left[T_O^pT_S^p - T_O^sT_S^s\right] + (1-y)\left[T_O^pT_S^s - T_O^sT_S^p\right]}{(1+y)\left[T_O^pT_S^p + T_O^sT_S^s\right] + (1-y)\left[T_O^pT_S^s + T_O^sT_S^p\right]}$$

$$T_{SyO} = T_ST_O\left(1+yD_SD_O\right) = 0.25\left\{(1+y)\left[T_O^pT_S^p + T_O^sT_S^s\right] + (1-y)\left[T_O^pT_S^s + T_O^sT_S^p\right]\right\}$$

$$y=+1 \Rightarrow D_{S+O} = \frac{T_O^pT_S^p - T_O^sT_S^s}{T_O^pT_S^p + T_O^sT_S^s}, \quad T_{S+O} = 0.5\left(T_O^pT_S^p + T_O^sT_S^s\right), \quad T_{S+O}^p = T_O^pT_S^p, \quad T_{S+O}^s = T_O^sT_S^s$$

$$y=-1 \Rightarrow D_{S\text{-}O} = \frac{T_O^pT_S^s - T_O^sT_S^p}{T_O^pT_S^s + T_O^sT_S^p}, \quad T_{S\text{-}O} = 0.5\left(T_O^pT_S^s + T_O^sT_S^p\right), \quad T_{S\text{-}O}^p = T_O^pT_S^s, \quad T_{S\text{-}O}^s = T_O^sT_S^p$$

(S.10.10.15)

$$D_S = \pm 1 \wedge y = \pm 1 \Rightarrow D_S^{\ 2} = y^2 = 1 \Rightarrow$$

$$D_{SyO} = \frac{D_O + yD_S}{1+yD_SD_O} = \frac{1}{yD_S}\frac{yD_SD_O + 1}{1+yD_SD_O} = \frac{1}{yD_S} = yD_S \tag{S.10.10.16}$$

$$1-D_{SyO} = \frac{T_{SyO}^s}{T_{SyO}}, 1+D_{SyO} = \frac{T_{SyO}^p}{T_{SyO}} \tag{S.10.10.17}$$

S.10.11 Retarder

A retarder is a retarding linear diattenuator (Sect. S.10.3ff) without diattenuation (see Chipman (2009b)):

$$D_O\text{=}0, Z_O\text{=}1 \text{ (without absorption } T_O = 1) \Rightarrow$$

$$M_{ret} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_O & s_O \\ 0 & 0 & -s_O & c_O \end{pmatrix}$$

(S.10.11.1

S.10.12 Rotated retarder

Rotated retarder with

$$
\begin{aligned}
&D_O = 0 \Rightarrow Z_O = \sqrt{1 - {D_O}^2} = 1,\ \ W_O = 1 - Z_O \mathrm{c}_O = 1 - \mathrm{c}_O \Rightarrow \\
&\mathbf{M}_{Ret}(\mathrm{x}\phi)/T_{Ret} = \mathbf{R}(\mathrm{x}\phi)\mathbf{M}_O\mathbf{R}(-\mathrm{x}\phi) = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \mathrm{s}_{2\phi}^2(1 - \mathrm{c}_O) & \mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}(1 - \mathrm{c}_O) & -\mathrm{xs}_{2\phi}\mathrm{s}_O \\ 0 & \mathrm{xs}_{2\phi}\mathrm{c}_{2\phi}(1 - \mathrm{c}_O) & 1 - \mathrm{c}_{2\phi}^2(1 - \mathrm{c}_O) & \mathrm{c}_{2\phi}\mathrm{s}_O \\ 0 & \mathrm{xs}_{2\phi}\mathrm{s}_O & -\mathrm{c}_{2\phi}\mathrm{s}_O & \mathrm{c}_O \end{pmatrix} \\
&\textit{with} \\
&1 - \mathrm{s}_{2\phi}^2(1 - \mathrm{c}_O) = \mathrm{c}_{2\phi}^2 + \mathrm{s}_{2\phi}^2\mathrm{c}_O \\
&1 - \mathrm{c}_{2\phi}^2(1 - \mathrm{c}_O) = \mathrm{s}_{2\phi}^2 + \mathrm{c}_{2\phi}^2\mathrm{c}_O
\end{aligned}
\qquad \text{(S.10.12.1)}
$$

S.10.13 Rotated λ/2 plate

Retarder Eq.(S.10.12.1) with

$$
\Delta_O = 180° \Rightarrow \mathrm{c}_O = -1, \mathrm{s}_O = 0, D_O = 0, Z_O = \sqrt{1 - {D_O}^2} = 1, W_O = 1 - Z_O\mathrm{c}_O = 2 \qquad \text{(S.10.13.1)}
$$

$$
\begin{aligned}
&\mathbf{M}_{HW}(\theta)/T_{HW} = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2\mathrm{s}_{2\theta}^2 & 2\mathrm{c}_{2\theta}\mathrm{s}_{2\theta} & 0 \\ 0 & 2\mathrm{c}_{2\theta}\mathrm{s}_{2\theta} & 1 - 2\mathrm{c}_{2\theta}^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{4\theta} & \mathrm{s}_{4\theta} & 0 \\ 0 & \mathrm{s}_{4\theta} & -\mathrm{c}_{4\theta} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{R}(2\theta)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
\qquad \text{(S.10.13.2)}
$$

The rotation of a λ/2 plate by ϕ rotates the Stokes vector by twice the rotation ϕ and additionally inverts the circular polarisation component. This is equivalent to a mirror followed by a rotation of the coordinate system by 2θ. Please note, that the rotator and mirror matrices don't commute (compare Eqs. (S.6.2.1) ff).

λ/2-retarder Eq.(S.10.12.1) at 0° and 22.5° with phase shift error 2ω

$$
\begin{aligned}
&D_O = 0 \Rightarrow Z_O = \sqrt{1 - {D_O}^2} = 1 \\
&\Delta_O = \pi + 2\omega \Rightarrow \mathrm{c}_O \approx -1 + 2\omega^2, \mathrm{s}_O \approx 2\omega \Rightarrow W_O = 1 - Z_O\mathrm{c}_O \approx 2 - 2\omega^2 \\
&\phi = 22.5° \Rightarrow \mathrm{s}_{2\phi} = \mathrm{c}_{2\phi} = \frac{1}{\sqrt{2}}
\end{aligned}
\qquad \text{(S.10.13.3)}
$$

$$\mathbf{M}_{HW}(0°,2\omega)/T_{HW} \approx \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1+2\omega^2 & 2\omega \\ 0 & 0 & -2\omega & -1+2\omega^2 \end{pmatrix} \tag{S.10.13.4}$$

$$\mathbf{M}_{HW}(\mathrm{x}22.5°,2\omega)/T_{HW} \approx \approx T_O \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & \mathrm{x}(1-\omega^2) & -\mathrm{x}\sqrt{2}\omega \\ 0 & \mathrm{x}(1-\omega^2) & \omega^2 & \sqrt{2}\omega \\ 0 & \mathrm{x}\sqrt{2}\omega & -\sqrt{2}\omega & -1+2\omega^2 \end{pmatrix} \tag{S.10.13.5}$$

$$\begin{aligned} &\left\langle \mathbf{A}_S(\mathrm{y}) \right| \mathbf{M}_{HW}(\mathrm{x}22.5°,2\omega) \approx \\ &\approx T_S \left\langle \begin{matrix} 1 \\ \mathrm{y}D_S \\ 0 \\ 0 \end{matrix} \right| \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 & \mathrm{x}(1-\omega^2) & -\mathrm{x}\sqrt{2}\omega \\ 0 & \mathrm{x}(1-\omega^2) & \omega^2 & \sqrt{2}\omega \\ 0 & \mathrm{x}\sqrt{2}\omega & -\sqrt{2}\omega & -1+2\omega^2 \end{pmatrix} \right| = \left\langle \begin{matrix} 1 \\ -\mathrm{y}\omega^2 D_S \\ -\mathrm{xy}(1-\omega^2) D_S \\ \mathrm{xy}\sqrt{2}\omega D_S \end{matrix} \right| = \\ &= \left\langle \begin{matrix} 1 \\ 0 \\ -\mathrm{xy}D_S \\ 0 \end{matrix} \right| + \mathrm{y}\omega D_S \left\langle \begin{matrix} 0 \\ -\omega \\ \mathrm{x}\omega \\ \mathrm{x}\sqrt{2} \end{matrix} \right| \end{aligned} \tag{S.10.13.6}$$

S.10.14 Rotated λ/2 plate for Δ90-calibration including error ε

$$\begin{aligned} &\frac{\mathbf{M}_{HW}(\mathrm{x}22.5° + \varepsilon/2)}{T_{HW}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{xs}_{2\varepsilon} & \mathrm{xc}_{2\varepsilon} & 0 \\ 0 & \mathrm{xc}_{2\varepsilon} & \mathrm{xs}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{xs}_{2\varepsilon} & -\mathrm{xc}_{2\varepsilon} & 0 \\ 0 & \mathrm{xc}_{2\varepsilon} & -\mathrm{xs}_{2\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{R}(\mathrm{x}45° + \varepsilon)\mathbf{M}_M \end{aligned} \tag{S.10.14.1}$$

S.10.15 Rotation calibrator

The mechanical rotator (Sect. S.10.2) and the λ/2- rotator (Sects. S.10.13, S.10.14) can be combined to

$$\frac{\mathbf{M}_{rot}(\phi,\mathrm{h})}{T_{rot}}=\mathbf{R}(\phi)\mathbf{M}_{\mathrm{h}}=\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{s}_{2\phi}&0\\0&\mathrm{s}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\mathrm{h}&0\\0&0&0&\mathrm{h}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{hs}_{2\phi}&0\\0&\mathrm{s}_{2\phi}&\mathrm{hc}_{2\phi}&0\\0&0&0&\mathrm{h}\end{pmatrix}=$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\mathrm{h}&0\\0&0&0&\mathrm{h}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\phi}&-\mathrm{hs}_{2\phi}&0\\0&\mathrm{hs}_{2\phi}&\mathrm{c}_{2\phi}&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\mathrm{h}&0\\0&0&0&\mathrm{h}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{\mathrm{h}2\phi}&-\mathrm{s}_{\mathrm{h}2\phi}&0\\0&\mathrm{s}_{\mathrm{h}2\phi}&\mathrm{c}_{\mathrm{h}2\phi}&0\\0&0&0&1\end{pmatrix}$$

$$\text{with } \mathrm{h}=\pm 1 \qquad \text{(S.10.15.1)}$$

where T_{rot} is the transmission of the rotation calibrator for unpolarised light, which equals one for the mechanical rotator. For the mechanical rotator we use h = +1, and for the λ/2- rotator h = −1, and $\phi = 2\theta$ is two times the actual rotation θ of the λ/2-plate, as well as ε is two times the actual error angle of the λ/2-plate. With Eq. (S.10.15.1) we get Eq. (S.10.15.2) for the rotation calibrator $\mathbf{M}_{rot}$ at ±45°.

$$\mathbf{M}_{rot}(\mathrm{x}45°+\varepsilon,\mathrm{h})/T_{rot}=\mathbf{R}(\mathrm{x}45°+\varepsilon)\mathbf{M}_{\mathrm{h}}=\mathbf{R}(\mathrm{x}45°)\mathbf{R}(\varepsilon)\mathbf{M}_{\mathrm{h}}=$$

$$=\begin{pmatrix}1&0&0&0\\0&0&-\mathrm{x}&0\\0&\mathrm{x}&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&\mathrm{c}_{2\varepsilon}&-\mathrm{s}_{2\varepsilon}&0\\0&\mathrm{s}_{2\varepsilon}&\mathrm{c}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\mathrm{h}&0\\0&0&0&\mathrm{h}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&-\mathrm{xhc}_{2\varepsilon}&0\\0&\mathrm{xc}_{2\varepsilon}&-\mathrm{xhs}_{2\varepsilon}&0\\0&0&0&\mathrm{h}\end{pmatrix} \qquad \text{(S.10.15.2)}$$

$$\text{with } \mathrm{x},\mathrm{h}=\pm 1$$

The error rotation ε can be separated as in Eq. (S.10.15.3) using the explanations in Sect. S.6.3 .

$$\mathbf{R}(\mathrm{x}45°+\varepsilon)\mathbf{M}_{\mathrm{h}}=\mathbf{R}(\mathrm{x}45°)\mathbf{R}(\varepsilon)\mathbf{M}_{\mathrm{h}}=\mathbf{R}(\mathrm{x}45°)\mathbf{M}_{\mathrm{h}}\mathbf{R}(\mathrm{h}\varepsilon) \qquad \text{(S.10.15.3)}$$

S.10.16 λ/4 plate (QWP)

From Eq. (S.10.6.1): QWP without diattenuation, with phase shift error ω.

$$\begin{aligned}
&\Delta_{QW}=90°+\omega \Rightarrow \mathrm{c}_Q=-\mathrm{s}_\omega,\ \mathrm{s}_Q=\mathrm{c}_\omega\\
&\phi=\mathrm{x}45°+\varepsilon \Rightarrow \mathrm{c}_{2\phi}\to -\mathrm{zs}_{2\varepsilon},\ \mathrm{s}_{2\phi}\to \mathrm{xc}_{2\varepsilon}\\
&D_{QW}=0, Z_{QW}=1, W_Q=\left(1-\mathrm{c}_{QW}\right)=\left(1+\mathrm{s}_\omega\right)\\
&\left(1-\mathrm{c}_{2\varepsilon}^2 W_{QW}\right)=\mathrm{s}_{2\varepsilon}^2+\mathrm{c}_{2\varepsilon}^2\mathrm{c}_{QW}=\left(\mathrm{s}_{2\varepsilon}^2-\mathrm{c}_{2\varepsilon}^2\mathrm{s}_\omega\right)\\
&\left(1-\mathrm{s}_{2\varepsilon}^2 W_{QW}\right)=\mathrm{c}_{2\varepsilon}^2+\mathrm{s}_{2\varepsilon}^2\mathrm{c}_{QW}=\left(\mathrm{c}_{2\varepsilon}^2-\mathrm{s}_{2\varepsilon}^2\mathrm{s}_\omega\right)
\end{aligned} \qquad \text{(S.10.16.1)}$$

$$D_{QW}=0, Z_{QW}=1, W_{QW}=\left(1-\mathrm{c}_{QW}\right)=\left(1+\mathrm{s}_{\omega}\right)\Rightarrow$$

1 (S.10.16.2)

$$\frac{\mathbf{M}_{QW}\left(\mathrm{x}45^{\circ}+\varepsilon,\omega\right)}{T_{QW}}=$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \left(1-\mathrm{c}_{2\varepsilon}^{2}W_{QW}\right) & -\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}W_{QW} & -\mathrm{xc}_{2\varepsilon}\mathrm{s}_{QW} \\ 0 & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}W_{QW} & \left(1-\mathrm{s}_{2\varepsilon}^{2}W_{QW}\right) & -\mathrm{xs}_{2\varepsilon}\mathrm{s}_{QW} \\ 0 & \mathrm{xc}_{2\varepsilon}\mathrm{s}_{QW} & \mathrm{xs}_{2\varepsilon}\mathrm{s}_{QW} & \mathrm{c}_{QW} \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{s}_{2\varepsilon}^{2}-\mathrm{c}_{2\varepsilon}^{2}\mathrm{s}_{\omega} & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}\left(1+\mathrm{s}_{\omega}\right) & -\mathrm{xc}_{2\varepsilon}\mathrm{c}_{\omega} \\ 0 & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}\left(1+\mathrm{s}_{\omega}\right) & \mathrm{c}_{2\varepsilon}^{2}-\mathrm{s}_{2\varepsilon}^{2}\mathrm{s}_{\omega} & -\mathrm{xs}_{2\varepsilon}\mathrm{c}_{\omega} \\ 0 & \mathrm{xc}_{2\varepsilon}\mathrm{c}_{\omega} & \mathrm{xs}_{2\varepsilon}\mathrm{c}_{\omega} & -\mathrm{s}_{\omega} \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\mathrm{c}_{2\varepsilon}^{2}\left(1+\mathrm{s}_{\omega}\right) & -\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\left(1+\mathrm{s}_{\omega}\right) & -\mathrm{xc}_{2\varepsilon}\mathrm{c}_{\omega} \\ 0 & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon}\left(1+\mathrm{s}_{\omega}\right) & 1-\mathrm{s}_{2\varepsilon}^{2}\left(1+\mathrm{s}_{\omega}\right) & -\mathrm{xs}_{2\varepsilon}\mathrm{c}_{\omega} \\ 0 & \mathrm{xc}_{2\varepsilon}\mathrm{c}_{\omega} & \mathrm{xs}_{2\varepsilon}\mathrm{c}_{\omega} & -\mathrm{s}_{\omega} \end{pmatrix}$$

$$D_{QW}=0, W_{Q}=\left(1+\mathrm{s}_{\omega}\right), \varepsilon=0\Rightarrow$$

2 (S.10.16.3)

$$\frac{\mathbf{M}_{QW}\left(\mathrm{x}45^{\circ},\omega\right)}{T_{QW}}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{s}_{\omega} & 0 & -\mathrm{xc}_{\omega} \\ 0 & 0 & 1 & 0 \\ 0 & \mathrm{xc}_{\omega} & 0 & -\mathrm{s}_{\omega} \end{pmatrix}$$

3 S.10.17 Rotated, ideal λ/4 plate

4 (S.10.17.1)

$$\Delta_{O}=90^{\circ}\Rightarrow \mathrm{c}_{O}=0,\ \ \mathrm{s}_{O}=1,\ \ D_{O}=0,\ \ Z_{O}=\sqrt{1-D_{O}^{\ 2}}=1,\ \ W_{O}=1-Z_{O}\mathrm{c}_{O}=1$$

(without absorption $T_{QWP}=1$)

5 (S.10.17.2)

$$\mathbf{M}_{QW}\left(\phi\right)=T_{QW}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}_{2\phi}^{2} & \mathrm{s}_{2\phi}\mathrm{c}_{2\phi} & -\mathrm{s}_{2\phi} \\ 0 & \mathrm{s}_{2\phi}\mathrm{c}_{2\phi} & \mathrm{s}_{2\phi}^{2} & \mathrm{c}_{2\phi} \\ 0 & \mathrm{s}_{2\phi} & -\mathrm{c}_{2\phi} & 0 \end{pmatrix}$$

6 (S.10.17.3)

$$\mathbf{M}_{QW}\left(\mathrm{x}45^{\circ}+\varepsilon\right)=\mathbf{R}\left(\mathrm{x}45^{\circ}+\varepsilon\right)\mathbf{M}_{QW}\mathbf{R}\left(-\mathrm{x}45^{\circ}-\varepsilon\right)=$$

$$=T_{QW}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{s}_{2\varepsilon}^{2} & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} & -\mathrm{xc}_{2\varepsilon} \\ 0 & -\mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon} & \mathrm{c}_{2\varepsilon}^{2} & -\mathrm{xs}_{2\varepsilon} \\ 0 & \mathrm{xc}_{2\varepsilon} & \mathrm{xs}_{2\varepsilon} & 0 \end{pmatrix}$$

$$\frac{\mathbf{M}_{QW}\left(\mathrm{x}45°\right)}{T_{QW}}=\begin{pmatrix}1&0&0&0\\0&0&0&-\mathrm{x}\\0&0&1&0\\0&\mathrm{x}&0&0\end{pmatrix} \quad \text{(S.10.17.4)}$$

$$\mathbf{M}_{QW}\left(0\right)=T_{QW}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix} \quad \text{(S.10.17.5)}$$

$\mathrm{x}\in\left\{0,\pm1\right\}$

$$\mathbf{M}_{QW}\left(\mathrm{x}45°\right)=T_{QW}\begin{pmatrix}1&0&0&0\\0&1-\mathrm{x}^2&0&-\mathrm{x}\\0&0&\mathrm{x}^2&1-\mathrm{x}^2\\0&\mathrm{x}&-\left(1-\mathrm{x}^2\right)&0\end{pmatrix} \quad \text{(S.10.17.6)}$$

$\mathrm{x}\in\left\{0,\pm1\right\}$

$$\mathbf{M}_{QW}\left(45°-\mathrm{x}45°\right)=T_{QW}\begin{pmatrix}1&0&0&0\\0&\mathrm{x}^2&0&-\left(1-\mathrm{x}^2\right)\\0&0&1-\mathrm{x}^2&\mathrm{x}\\0&1-\mathrm{x}^2&-\mathrm{x}&0\end{pmatrix} \quad \text{(S.10.17.7)}$$

S.10.18 Circular polariser

Linear polariser at 0° Eq. (S.10.3.1) and QWP at z45° Eq. (S.10.16.3) (see Chipman (2009a) Chap. 15.26).

$D_{QW}=0,\ Z_{QW}=1, \Delta_{QW}=90°+\omega\ \Rightarrow W_{QW}=\left(1+\mathrm{s}_\omega\right)=$

$$\frac{\mathbf{M}_{QW}\left(\mathrm{z}45°,\omega\right)\mathbf{M}_P\left(0°\right)}{T_{QW}T_P}=\begin{pmatrix}1&0&0&0\\0&-\mathrm{s}_\omega&0&-\mathrm{zc}_\omega\\0&0&1&0\\0&\mathrm{zc}_\omega&0&-\mathrm{s}_\omega\end{pmatrix}\begin{pmatrix}1&D_P&0&0\\D_P&1&0&0\\0&0&Z_P\mathrm{c}_P&Z_P\mathrm{s}_P\\0&0&-Z_P\mathrm{s}_P&Z_P\mathrm{c}_P\end{pmatrix}= \quad \text{(S.10.18.1)}$$

$$=\begin{pmatrix}1&D_P&0&0\\-\mathrm{s}_\omega D_P&-\mathrm{s}_\omega&\mathrm{zc}_\omega Z_P\mathrm{s}_P&-\mathrm{zc}_\omega Z_P\mathrm{c}_P\\0&0&Z_P\mathrm{c}_P&Z_P\mathrm{s}_P\\\mathrm{zc}_\omega D_P&\mathrm{zc}_\omega&\mathrm{s}_\omega Z_P\mathrm{s}_P&-\mathrm{s}_\omega Z_P\mathrm{c}_P\end{pmatrix}$$

Circular polariser as ±45° calibrator with a QWP with phase shift error ω and a real linear polariser

$$\frac{\left[\mathbf{M}_{QW}(\mathrm{z}45^\circ,\omega)\mathbf{M}_P(0^\circ)\right](\mathrm{x}45^\circ+\varepsilon)}{T_{QW}T_P}=$$

$$=\begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon}D_P & \ldots & \ldots \\ \mathrm{xs}_{2\varepsilon}\mathrm{s}_\omega D_P & -\mathrm{s}_{2\varepsilon}^2\mathrm{s}_\omega+\mathrm{c}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P+\mathrm{s}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{s}_P\right)Z_P & \ldots & \ldots \\ -\mathrm{xc}_{2\varepsilon}\mathrm{s}_\omega D_P & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega+\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P-\mathrm{c}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{s}_P\right)Z_P & \ldots & \ldots \\ \mathrm{zc}_\omega D_P & -\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{zc}_\omega+\mathrm{c}_{2\varepsilon}\mathrm{s}_\omega\mathrm{s}_P Z_P\right) & \ldots & \ldots \\ \ldots & \ldots & \mathrm{xc}_{2\varepsilon}D_P & 0 \\ \ldots & \ldots & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega+\mathrm{s}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P+\mathrm{s}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{s}_P\right)Z_P & \mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{c}_P-\mathrm{c}_{2\varepsilon}\mathrm{s}_P\right)Z_P \\ \ldots & \ldots & -\mathrm{c}_{2\varepsilon}^2\mathrm{s}_\omega+\mathrm{s}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P-\mathrm{c}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{s}_P\right)Z_P & -\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{zc}_\omega\mathrm{c}_P+\mathrm{s}_{2\varepsilon}\mathrm{s}_P\right)Z_P \\ \ldots & \ldots & \mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{zc}_\omega-\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega\mathrm{s}_P Z_P\right) & -\mathrm{s}_\omega\mathrm{c}_P Z_P \end{pmatrix} \qquad \text{(S.10.18.2)}$$

Circular polariser with QWP without phase shift error ω and real linear polariser as ±45° calibrator

$$\omega=0\ \Rightarrow$$

$$\frac{\mathbf{M}_{CP}(\mathrm{z},0,\mathrm{x}45^\circ+\varepsilon)}{T_{CP}}=\frac{\left[\mathbf{M}_{QW}(\mathrm{z}45^\circ,0)\mathbf{M}_P(0^\circ)\right](\mathrm{x}45^\circ+\varepsilon)}{T_{QW}T_P}=$$

$$=\begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon}D_P & \mathrm{xc}_{2\varepsilon}D_P & 0 \\ 0 & \mathrm{c}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P+\mathrm{zs}_{2\varepsilon}\mathrm{s}_P\right)Z_P & \mathrm{s}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P+\mathrm{zs}_{2\varepsilon}\mathrm{s}_P\right)Z_P & \mathrm{x}\left(\mathrm{zs}_{2\varepsilon}\mathrm{c}_P-\mathrm{c}_{2\varepsilon}\mathrm{s}_P\right)Z_P \\ 0 & \mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P-\mathrm{zc}_{2\varepsilon}\mathrm{s}_P\right)Z_P & \mathrm{s}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P-\mathrm{zc}_{2\varepsilon}\mathrm{s}_P\right)Z_P & -\mathrm{x}\left(\mathrm{zc}_{2\varepsilon}\mathrm{c}_P+\mathrm{s}_{2\varepsilon}\mathrm{s}_P\right)Z_P \\ \mathrm{z}D_P & -\mathrm{xzs}_{2\varepsilon} & \mathrm{xzc}_{2\varepsilon} & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon}D_P & \mathrm{xc}_{2\varepsilon}D_P & 0 \\ 0 & \mathrm{c}_{2\varepsilon}\mathrm{c}_{2\varepsilon-\mathrm{z}P}Z_P & \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon-\mathrm{z}P}Z_P & \mathrm{xs}_{\mathrm{z}2\varepsilon-P}Z_P \\ 0 & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon-\mathrm{z}P}Z_P & \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon-\mathrm{z}P}Z_P & -\mathrm{xc}_{\mathrm{z}2\varepsilon-P}Z_P \\ \mathrm{z}D_P & -\mathrm{xzs}_{2\varepsilon} & \mathrm{xzc}_{2\varepsilon} & 0 \end{pmatrix}$$

(S.10.18.3)

Circular polariser with QWP with phase shift error ω and ideal linear polariser as ±45° calibrator

$$D_P=1,\ Z_P=0,\ \Rightarrow$$

$$\frac{\mathbf{M}_{CP}(\mathrm{z},\omega,\mathrm{x}45^\circ+\varepsilon)}{T_{CP}}=\frac{\left[\mathbf{M}_{QW}(\mathrm{z}45^\circ,\omega)\mathbf{M}_P(0^\circ)\right](\mathrm{x}45^\circ+\varepsilon)}{T_{QW}T_P}=$$

$$=\begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon} & \mathrm{xc}_{2\varepsilon} & 0 \\ \mathrm{xs}_{2\varepsilon}\mathrm{s}_\omega & -\mathrm{s}_{2\varepsilon}^2\mathrm{s}_\omega & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega & 0 \\ -\mathrm{xc}_{2\varepsilon}\mathrm{s}_\omega & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega & -\mathrm{c}_{2\varepsilon}^2\mathrm{s}_\omega & 0 \\ \mathrm{zc}_\omega & -\mathrm{xs}_{2\varepsilon}\mathrm{zc}_\omega & \mathrm{xc}_{2\varepsilon}\mathrm{zc}_\omega & 0 \end{pmatrix}=\left|\begin{matrix} 1 \\ \mathrm{xs}_{2\varepsilon}\mathrm{s}_\omega \\ -\mathrm{xc}_{2\varepsilon}\mathrm{s}_\omega \\ \mathrm{zc}_\omega \end{matrix}\right\rangle\left\langle\begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix}\right| \qquad \text{(S.10.18.4)}$$

Ideal circular polariser as ±45° calibrator (see Eq. (E.27)) with $D_P = 1,\ \omega = 0 \Rightarrow$

$\mathbf{M}_{CP}(\mathrm{z}) = \mathbf{M}_{QW}(\mathrm{z}45°)\mathbf{M}_P(0°) =$

1 $$= T_{CP}\begin{pmatrix}1&0&0&0\\0&0&0&-\mathrm{z}\\0&0&1&0\\0&\mathrm{z}&0&0\end{pmatrix}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} = T_{CP}\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\\\mathrm{z}&\mathrm{z}&0&0\end{pmatrix} \tag{S.10.18.5}$$

$$= T_{CP}\begin{pmatrix}1&0&0&0\\0&0&0&-\mathrm{z}\\0&0&1&0\\0&\mathrm{z}&0&0\end{pmatrix}\left|\begin{matrix}1\\1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right| = T_{CP}\left|\begin{matrix}1\\0\\0\\\mathrm{z}\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right|$$

with $\mathrm{z} = \pm 1$

2 Rotated, ideal circular polariser as ±45° calibrator

$D_P = 1,\ \omega = 0\ \Rightarrow$

$$\frac{\mathbf{M}_{CP}(\mathrm{z},\mathrm{x}45°+\varepsilon)}{T_{CP}} = \frac{\mathbf{R}(\mathrm{x}45°+\varepsilon)\mathbf{M}_{QW}(\mathrm{z}45°)\mathbf{M}_P(0°)\mathbf{R}(-\mathrm{x}45°-\varepsilon)}{T_{CP}} =$$

3 $$= \begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&-\mathrm{xc}_{2\varepsilon}&0\\0&\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&0&-\mathrm{z}\\0&0&1&0\\0&\mathrm{z}&0&0\end{pmatrix}\left|\begin{matrix}1\\1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right|\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\0&-\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix} = \tag{S.10.18.6}$$

$$= \begin{pmatrix}1&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\0&0&0&0\\0&0&0&0\\\mathrm{z}&-\mathrm{zxs}_{2\varepsilon}&\mathrm{zxc}_{2\varepsilon}&0\end{pmatrix} = \left|\begin{matrix}1\\0\\0\\\mathrm{z}\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right|$$

Rotated circular polariser as +-45° calibrator with a QWP, with retardation error

From Eq. (S.10.9.3)

$$\frac{\mathbf{M}_{QW}(\phi)\mathbf{M}_P}{T_{QW}T_O} =$$

$$= \begin{pmatrix}
1+c_{2\phi}D_{QW}D_P & c_{2\phi}D_{QW}+D_P & .\ . \\
c_{2\phi}D_{QW}+\left(1-s^2_{2\phi}W_{QW}\right)D_P & c_{2\phi}D_{QW}D_P+\left(1-s^2_{2\phi}W_{QW}\right) & .\ . \\
s_{2\phi}\left(D_{QW}+c_{2\phi}W_{QW}D_P\right) & s_{2\phi}\left(D_{QW}D_P+c_{2\phi}W_{QW}\right) & .\ . \\
s_{2\phi}Z_{QW}s_{QW}D_P & s_{2\phi}Z_{QW}s_{QW} & .\ . \\
.\ . & s_{2\phi}D_{QW}Z_Pc_P & s_{2\phi}D_{QW}Z_Ps_P \\
.\ . & s_{2\phi}\left(c_{2\phi}W_{QW}c_P+Z_{QW}s_{QW}s_P\right)Z_P & s_{2\phi}\left(c_{2\phi}W_{QW}s_P-Z_{QW}s_{QW}c_P\right)Z_P \\
.\ . & \left[\left(1-c^2_{2\phi}W_{QW}\right)c_P-c_{2\phi}Z_{QW}s_{QW}s_P\right]Z_P & \left[\left(1-c^2_{2\phi}W_{QW}\right)s_P+c_{2\phi}Z_{QW}s_{QW}c_P\right]Z_P \\
.\ . & -\left(c_{2\phi}s_{QW}c_P+c_{QW}s_P\right)Z_{QW}Z_P & \left(-c_{2\phi}s_{QW}s_P+c_{QW}c_P\right)Z_{QW}Z_P
\end{pmatrix} \quad \text{(S.10.18.7)}$$

$$\mathbf{M}_{QW}(\phi)\mathbf{M}_P = T_{QW}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^2_{2\phi} & s_{2\phi}c_{2\phi} & -s_{2\phi} \\ 0 & s_{2\phi}c_{2\phi} & s^2_{2\phi} & c_{2\phi} \\ 0 & s_{2\phi} & -c_{2\phi} & 0 \end{pmatrix}\left|\begin{matrix}1\\1\\0\\0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right| = T_{QW}\left|\begin{matrix}1\\c^2_{2\phi}\\s_{2\phi}c_{2\phi}\\s_{2\phi}\end{matrix}\right\rangle\left\langle\begin{matrix}1\\1\\0\\0\end{matrix}\right| \quad \text{(S.10.18.8)}$$

$$\phi = \mathrm{x}45^\circ+\varepsilon \Rightarrow c_{2\phi}\to -\mathrm{x}s_{2\varepsilon}, s_{2\phi}\to \mathrm{x}c_{2\varepsilon} \Rightarrow$$

$$\frac{\mathbf{M}_{QW}(\mathrm{x}45^\circ+\varepsilon)\mathbf{M}_P}{T_{QW}T_P} =$$

$$= \begin{pmatrix}
1-\mathrm{x}s_{2\varepsilon}D_{QW}D_P & -\mathrm{x}s_{2\varepsilon}D_{QW}+D_P & .\ . \\
-\mathrm{x}s_{2\varepsilon}D_{QW}+\left(1-c^2_{2\varepsilon}W_{QW}\right)D_P & -\mathrm{x}s_{2\varepsilon}D_{QW}D_P+\left(1-c^2_{2\varepsilon}W_{QW}\right) & .\ . \\
c_{2\varepsilon}\left(\mathrm{x}D_{QW}-s_{2\varepsilon}W_{QW}D_P\right) & c_{2\varepsilon}\left(\mathrm{x}D_{QW}D_P-s_{2\varepsilon}W_{QW}\right) & .\ . \\
\mathrm{x}c_{2\varepsilon}Z_{QW}s_{QW}D_P & \mathrm{x}c_{2\varepsilon}Z_{QW}s_{QW} & .\ . \\
.\ . & \mathrm{x}c_{2\varepsilon}D_{QW}Z_Pc_P & \mathrm{x}c_{2\varepsilon}D_{QW}Z_Ps_P \\
.\ . & -c_{2\varepsilon}\left(s_{2\varepsilon}W_{QW}c_P-\mathrm{x}Z_{QW}s_{QW}s_P\right)Z_P & -c_{2\varepsilon}\left(s_{2\varepsilon}W_{QW}s_P+\mathrm{x}Z_{QW}s_{QW}c_P\right)Z_P \\
.\ . & \left[\left(1-s^2_{2\varepsilon}W_{QW}\right)c_P+\mathrm{x}s_{2\varepsilon}Z_{QW}s_{QW}s_P\right]Z_P & \left[\left(1-s^2_{2\varepsilon}W_{QW}\right)s_P-\mathrm{x}s_{2\varepsilon}Z_{QW}s_{QW}c_P\right]Z_P \\
.\ . & -\left(-\mathrm{x}s_{2\varepsilon}s_{QW}c_P+c_{QW}s_P\right)Z_{QW}Z_P & \left(+\mathrm{x}s_{2\varepsilon}s_{QW}s_P+c_{QW}c_P\right)Z_{QW}Z_P
\end{pmatrix}$$

(S.10.18.9)

$$\phi = 45^\circ + \varepsilon \Rightarrow \mathrm{c}_{2\phi} \to -\mathrm{s}_{2\varepsilon}, \mathrm{s}_{2\phi} \to \mathrm{c}_{2\varepsilon} \Rightarrow$$

$$\frac{\mathbf{M}_{QW}\left(45^\circ + \varepsilon\right)\mathbf{M}_P}{T_{QW}T_P} =$$

$$= \begin{pmatrix} 1 - \mathrm{s}_{2\varepsilon} D_{QW} D_P & -\mathrm{s}_{2\varepsilon} D_{QW} + D_P & \cdot & \cdot \\ -\mathrm{s}_{2\varepsilon} D_{QW} + \left(1 - \mathrm{c}_{2\varepsilon}^2 W_{QW}\right) D_P & -\mathrm{s}_{2\varepsilon} D_{QW} D_P + \left(1 - \mathrm{c}_{2\varepsilon}^2 W_{QW}\right) & \cdot & \cdot \\ \mathrm{c}_{2\varepsilon}\left(D_{QW} - \mathrm{s}_{2\varepsilon} W_{QW} D_P\right) & \mathrm{c}_{2\varepsilon}\left(D_{QW} D_P - \mathrm{s}_{2\varepsilon} W_{QW}\right) & \cdot & \cdot \\ \mathrm{c}_{2\varepsilon} Z_{QW} \mathrm{s}_{QW} D_P & \mathrm{c}_{2\varepsilon} Z_{QW} \mathrm{s}_{QW} & \cdot & \cdot \\ \cdot & \cdot & \mathrm{c}_{2\varepsilon} D_{QW} Z_P \mathrm{c}_P & \mathrm{c}_{2\varepsilon} D_{QW} Z_P \mathrm{s}_P \\ \cdot & \cdot & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon} W_{QW} \mathrm{c}_P - Z_{QW} \mathrm{s}_{QW} \mathrm{s}_P\right) Z_P & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon} W_{QW} \mathrm{s}_P + Z_{QW} \mathrm{s}_{QW} \mathrm{c}_P\right) Z_P \\ \cdot & \cdot & \left[\left(1 - \mathrm{s}_{2\varepsilon}^2 W_{QW}\right)\mathrm{c}_P + \mathrm{s}_{2\varepsilon} Z_{QW} \mathrm{s}_{QW} \mathrm{s}_P\right] Z_P & \left[\left(1 - \mathrm{s}_{2\varepsilon}^2 W_{QW}\right)\mathrm{s}_P - \mathrm{s}_{2\varepsilon} Z_{QW} \mathrm{s}_{QW} \mathrm{c}_P\right] Z_P \\ \cdot & \cdot & \left(\mathrm{s}_{2\varepsilon} \mathrm{s}_{QW} \mathrm{c}_P - \mathrm{c}_{QW} \mathrm{s}_P\right) Z_{QW} Z_P & \left(\mathrm{s}_{2\varepsilon} \mathrm{s}_{QW} \mathrm{s}_P + \mathrm{c}_{QW} \mathrm{c}_P\right) Z_{QW} Z_P \end{pmatrix}$$

1 (S.10.18.10)

$$D_Q = 0, \mathrm{c}_{QW} = 0, \mathrm{s}_{QW} = 1, Z_{QW} = 1, W_{QW} = \left(1 - \mathrm{c}_{QW}\right) = 1 \Rightarrow$$

$$\frac{\left[\mathbf{M}_{QW}\left(45^\circ + \varepsilon\right)\mathbf{M}_P\right]\left(\mathrm{x}45^\circ\right)}{T_{QW}T_P} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{x} & 0 \\ 0 & -\mathrm{x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & D_P & \cdot & \cdot \\ \mathrm{s}_{2\varepsilon}^2 D_P & \mathrm{s}_{2\varepsilon}^2 & \cdot & \cdot \\ -\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon} D_P & -\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon} D_P & \cdot & \cdot \\ \mathrm{c}_{2\varepsilon} D_P & \mathrm{c}_{2\varepsilon} & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P - \mathrm{s}_P\right) Z_P & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_P + \mathrm{c}_P\right) Z_P \\ \cdot & \cdot & \left(\mathrm{c}_{2\varepsilon}^2 \mathrm{c}_P + \mathrm{s}_{2\varepsilon}\mathrm{s}_P\right) Z_P & \left(\mathrm{c}_{2\varepsilon}^2 \mathrm{s}_P - \mathrm{s}_{2\varepsilon}\mathrm{c}_P\right) Z_P \\ \cdot & \cdot & \mathrm{s}_{2\varepsilon}\mathrm{c}_P Z_P & \mathrm{s}_{2\varepsilon}\mathrm{s}_P Z_P \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{x} & 0 \\ 0 & \mathrm{x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & -\mathrm{x} D_P & 0 \\ \mathrm{s}_{2\varepsilon}^2 D_P & \left(\mathrm{c}_{2\varepsilon}^2 \mathrm{c}_P + \mathrm{s}_{2\varepsilon}\mathrm{s}_P\right) Z_P & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon} D_P & \mathrm{x}\left(\mathrm{c}_{2\varepsilon}^2 \mathrm{s}_P - \mathrm{s}_{2\varepsilon}\mathrm{c}_P\right) Z_P \\ -\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon} D_P & \mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P - \mathrm{s}_P\right) Z_P & \mathrm{s}_{2\varepsilon}^2 & -\mathrm{x}\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_P + \mathrm{c}_P\right) Z_P \\ \mathrm{c}_{2\varepsilon} D_P & \mathrm{x}\mathrm{s}_{2\varepsilon}\mathrm{c}_P Z_P & -\mathrm{x}\mathrm{c}_{2\varepsilon} & \mathrm{s}_{2\varepsilon}\mathrm{s}_P Z_P \end{pmatrix}$$

2 (S.10.18.11)

$$\left.\begin{aligned}
&\phi = 45^\circ + \varepsilon \Rightarrow \mathrm{c}_{2\phi} \to -\mathrm{s}_{2\varepsilon}, \mathrm{s}_{2\phi} \to \mathrm{c}_{2\varepsilon}, \mathit{\Delta}_{QW}{=}90^\circ{+}\varsigma \Rightarrow \ \mathrm{c}_{QW} = -\mathrm{s}_\varsigma \to 0,\ \ \mathrm{s}_{QW} = \mathrm{c}_\varsigma \to 1\\
&\left(1-\mathrm{c}_{2\varepsilon}^2 W_{QW}\right) = \left(1-\mathrm{c}_{2\varepsilon}^2\left(1-Z_{QW}\mathrm{c}_{QW}\right)\right) = \mathrm{s}_{2\varepsilon}^2 + \mathrm{c}_{2\varepsilon}^2 Z_{QW}\mathrm{c}_{QW} = \left(\mathrm{s}_{2\varepsilon}^2 - \mathrm{c}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right),\\
&\left(1-\mathrm{s}_{2\varepsilon}^2 W_{QW}\right) = \mathrm{c}_{2\varepsilon}^2 + \mathrm{s}_{2\varepsilon}^2 Z_Q\mathrm{c}_{QW} = \left(\mathrm{c}_{2\varepsilon}^2 - \mathrm{s}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right)
\end{aligned}\right\}\Rightarrow$$

$$\frac{\left[\mathbf{M}_{QW}\left(45^\circ+\varepsilon, \mathit{\Delta}_{QW}{=}90^\circ+\varsigma\right)\mathbf{M}_P\right]}{T_{QW}T_P} =$$

$$= \begin{pmatrix}
1-\mathrm{s}_{2\varepsilon}D_{QW}D_P & -\mathrm{s}_{2\varepsilon}D_{QW}+D_P & \cdot & \cdot\\
-\mathrm{s}_{2\varepsilon}D_{QW}+\left(\mathrm{s}_{2\varepsilon}^2-\mathrm{c}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right)D_P & -\mathrm{s}_{2\varepsilon}D_{QW}D_P+\left(\mathrm{s}_{2\varepsilon}^2-\mathrm{c}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right) & \cdot & \cdot\\
\mathrm{c}_{2\varepsilon}\left(D_{QW}-\mathrm{s}_{2\varepsilon}\left(1-Z_{QW}\mathrm{c}_{QW}\right)D_P\right) & \mathrm{c}_{2\varepsilon}\left(D_{QW}D_P-\mathrm{s}_{2\varepsilon}\left(1-Z_{QW}\mathrm{c}_{QW}\right)\right) & \cdot & \cdot\\
\mathrm{c}_{2\varepsilon}Z_{QW}\mathrm{c}_\varsigma D_P & \mathrm{c}_{2\varepsilon}Z_{QW}\mathrm{c}_\varsigma & \cdot & \cdot\\
\cdot & \cdot & \mathrm{c}_{2\varepsilon}D_{QW}Z_P\mathrm{c}_P & \mathrm{c}_{2\varepsilon}D_{QW}Z_P\mathrm{s}_P\\
\cdot & \cdot & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\left(1-Z_{QW}\mathrm{c}_{QW}\right)\mathrm{c}_P - Z_{QW}\mathrm{c}_\varsigma\mathrm{s}_P\right)Z_P & -\mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\left(1-Z_{QW}\mathrm{c}_{QW}\right)\mathrm{s}_P + Z_{QW}\mathrm{c}_\varsigma\mathrm{c}_P\right)Z_P\\
\cdot & \cdot & \left[\left(\mathrm{c}_{2\varepsilon}^2-\mathrm{s}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right)\mathrm{c}_P+\mathrm{s}_{2\varepsilon}Z_{QW}\mathrm{c}_\varsigma\mathrm{s}_P\right]Z_P & \left[\left(\mathrm{c}_{2\varepsilon}^2-\mathrm{s}_{2\varepsilon}^2 Z_{QW}\mathrm{s}_\varsigma\right)\mathrm{s}_P-\mathrm{s}_{2\varepsilon}Z_{QW}\mathrm{c}_\varsigma\mathrm{c}_P\right]Z_P\\
\cdot & \cdot & \left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma\mathrm{c}_P+\mathrm{s}_\varsigma\mathrm{s}_P\right)Z_{QW}Z_P & \left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma\mathrm{s}_P-\mathrm{s}_\varsigma\mathrm{c}_P\right)Z_{QW}Z_P
\end{pmatrix}$$

1 (S.10.18.12)

$$\phi = 45^\circ \Rightarrow \mathrm{c}_{2\phi} = 0, \mathrm{s}_{2\phi} = 1 \Rightarrow$$

2 $$\frac{\mathbf{M}_{QW}\left(45^\circ\right)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix}
1 & D_P & \mathrm{c}_P D_{QW}Z_P & \mathrm{s}_P D_{QW}Z_P\\
\mathrm{c}_{QW}Z_{QW}D_P & \mathrm{c}_{QW}Z_{QW} & \mathrm{s}_{QW}\mathrm{s}_P Z_{QW}Z_P & -\mathrm{s}_{QW}\mathrm{c}_P Z_{QW}Z_P\\
D_{QW} & D_{QW}D_P & \mathrm{c}_P Z_P & \mathrm{s}_P Z_P\\
\mathrm{s}_{QW}Z_{QW}D_P & Z_{QW}\mathrm{s}_{QW} & -\mathrm{c}_{QW}\mathrm{s}_P Z_{QW}Z_P & \mathrm{c}_{QW}\mathrm{c}_P Z_{QW}Z_P
\end{pmatrix}$$ (S.10.18.13)

$$D_{QW} = 0, Z_{QW} = 1, W_{QW} = \left(1-\mathrm{c}_{QW}\right), \phi = 45^\circ + \varepsilon \Rightarrow \mathrm{c}_{2\phi} \to -\mathrm{s}_{2\varepsilon}, \mathrm{s}_{2\phi} \to \mathrm{c}_{2\varepsilon} \Rightarrow$$

3 $$\frac{\mathbf{M}_{CP}\left(0^\circ\right)}{T_{CP}} = \frac{\mathbf{M}_{QW}\left(45^\circ\right)\mathbf{M}_P}{T_{QW}T_P} = \begin{pmatrix}
1 & D_P & 0 & 0\\
\mathrm{c}_{QW}D_P & \mathrm{c}_{QW} & \mathrm{s}_{QW}\mathrm{s}_P Z_P & -\mathrm{s}_{QW}\mathrm{c}_P Z_P\\
0 & 0 & \mathrm{c}_P Z_P & \mathrm{s}_P Z_P\\
\mathrm{s}_{QW}D_P & \mathrm{s}_{QW} & -\mathrm{c}_{QW}\mathrm{s}_P Z_P & \mathrm{c}_{QW}\mathrm{c}_P Z_P
\end{pmatrix}$$ (S.10.18.14)

$$\phi = 45^\circ + \varepsilon \Rightarrow \mathrm{c}_{2\phi} \to -\mathrm{s}_{2\varepsilon}, \mathrm{s}_{2\phi} \to \mathrm{c}_{2\varepsilon}, \mathit{\Delta}_{QW}{=}90^\circ{+}\varsigma \Rightarrow \ \mathrm{c}_{QW} = -\mathrm{s}_\varsigma,\ \ \mathrm{s}_{QW} = \mathrm{c}_\varsigma \Rightarrow$$

$$\frac{\left[\mathbf{M}_{QW}\left(45^\circ, \mathit{\Delta}_{QW}{=}90^\circ+\varsigma\right)\mathbf{M}_P\right]}{T_{QW}T_P} = \begin{pmatrix}
1 & D_P & \mathrm{c}_P D_{QW}Z_P & \mathrm{s}_P D_{QW}Z_P\\
-\mathrm{s}_\varsigma Z_{QW}D_P & -\mathrm{s}_\varsigma Z_{QW} & \mathrm{c}_\varsigma\mathrm{s}_P Z_{QW}Z_P & -\mathrm{c}_\varsigma\mathrm{c}_P Z_{QW}Z_P\\
D_{QW} & D_{QW}D_P & \mathrm{c}_P Z_P & \mathrm{s}_P Z_P\\
\mathrm{c}_\varsigma Z_{QW}D_P & \mathrm{c}_\varsigma Z_{QW} & \mathrm{s}_\varsigma\mathrm{s}_P Z_{QW}Z_P & -\mathrm{s}_\varsigma\mathrm{c}_P Z_{QW}Z_P
\end{pmatrix}$$

4 (S.10.18.15)

$$\frac{\left[\mathbf{M}_{QW}\left(45°,\varDelta_Q{=}90°+\varsigma\right)\mathbf{M}_P\right]\left(\mathrm{x}45°+\varepsilon\right)}{T_{QW}T_P}=$$

$$=\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&-\mathrm{xc}_{2\varepsilon}&0\\0&\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&D_P&\mathrm{c}_PD_{QW}Z_P&\mathrm{s}_PD_{QW}Z_P\\-\mathrm{s}_\varsigma Z_{QW}D_P&-\mathrm{s}_\varsigma Z_Q&\mathrm{c}_\varsigma\mathrm{s}_PZ_{QW}Z_P&-\mathrm{c}_\varsigma\mathrm{c}_PZ_{QW}Z_P\\D_{QW}&D_{QW}D_P&\mathrm{c}_PZ_P&\mathrm{s}_PZ_P\\\mathrm{c}_\varsigma Z_{QW}D_P&\mathrm{c}_\varsigma Z_Q&\mathrm{s}_\varsigma\mathrm{s}_PZ_{QW}Z_P&-\mathrm{s}_\varsigma\mathrm{c}_PZ_{QW}Z_P\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\0&-\mathrm{xc}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\0&0&0&1\end{pmatrix}=$$

$$=\begin{pmatrix}1&-\mathrm{x}\left(\mathrm{s}_{2\varepsilon}D_P+\mathrm{c}_{2\varepsilon}\mathrm{c}_PD_QZ_P\right)&..\\\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}D_P-\mathrm{c}_{2\varepsilon}D_{QW}\right)&-\mathrm{s}_{2\varepsilon}^2\mathrm{s}_\varsigma Z_{QW}+\mathrm{c}_{2\varepsilon}^2\mathrm{c}_PZ_P+\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\left(\mathrm{c}_\varsigma\mathrm{s}_PZ_PZ_{QW}+D_{QW}D_P\right)&..\\-\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}D_P+\mathrm{s}_{2\varepsilon}D_{QW}\right)&\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\left(\mathrm{s}_\varsigma Z_{QW}+\mathrm{c}_PZ_P\right)-\mathrm{c}_{2\varepsilon}^2\mathrm{c}_\varsigma\mathrm{s}_PZ_PZ_{QW}+\mathrm{s}_{2\varepsilon}^2D_{QW}D_P&..\\\mathrm{c}_\varsigma Z_{QW}D_P&-\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma+\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma\mathrm{s}_PZ_P\right)Z_{QW}&..\\..&\mathrm{x}\left(\mathrm{c}_{2\varepsilon}D_P-\mathrm{s}_{2\varepsilon}\mathrm{c}_PD_{QW}Z_P\right)&\mathrm{s}_PD_{QW}Z_P\\..&\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\left(\mathrm{s}_\varsigma Z_{QW}+\mathrm{c}_PZ_P\right)+\mathrm{s}_{2\varepsilon}^2\mathrm{c}_\varsigma\mathrm{s}_PZ_PZ_{QW}-\mathrm{c}_{2\varepsilon}^2D_{QW}D_P&\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma\mathrm{c}_PZ_{QW}-\mathrm{c}_{2\varepsilon}\mathrm{s}_P\right)Z_P\\..&-\mathrm{c}_{2\varepsilon}^2\mathrm{s}_\varsigma Z_{QW}+\mathrm{s}_{2\varepsilon}^2\mathrm{c}_PZ_P-\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\left(\mathrm{c}_\varsigma\mathrm{s}_PZ_PZ_{QW}+D_{QW}D_P\right)&-\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma\mathrm{c}_PZ_{QW}+\mathrm{s}_{2\varepsilon}\mathrm{s}_P\right)Z_P\\..&\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma-\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma\mathrm{s}_PZ_P\right)Z_{QW}&-\mathrm{s}_\varsigma\mathrm{c}_PZ_{QW}Z_P\end{pmatrix}$$

1 (S.10.18.16)

CP with ideal LP: $D_P=1,\ \ Z_P=0,\ \ \Rightarrow$

$$\frac{\left[\mathbf{M}_{QW}\left(45°,\varDelta_{QW}{=}90°+\varsigma\right)\mathbf{M}_P\right]\left(\mathrm{x}45°+\varepsilon\right)}{T_{QW}T_P}=$$

$$=\begin{pmatrix}1&-\mathrm{xs}_{2\varepsilon}&\mathrm{xc}_{2\varepsilon}&0\\\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}-\mathrm{c}_{2\varepsilon}D_{QW}\right)&-\mathrm{s}_{2\varepsilon}^2\mathrm{s}_\varsigma Z_{QW}+\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}D_{QW}&\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}-\mathrm{c}_{2\varepsilon}^2D_{QW}&0\\-\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}+\mathrm{s}_{2\varepsilon}D_{QW}\right)&\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}+\mathrm{s}_{2\varepsilon}^2D_{QW}&-\mathrm{c}_{2\varepsilon}^2\mathrm{s}_\varsigma Z_{QW}-\mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}D_{QW}&0\\\mathrm{c}_\varsigma Z_{QW}&-\mathrm{xs}_{2\varepsilon}\mathrm{c}_\varsigma Z_{QW}&\mathrm{xc}_{2\varepsilon}\mathrm{c}_\varsigma Z_{QW}&0\end{pmatrix}=$$

$$=\left|\begin{matrix}1\\\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}-\mathrm{c}_{2\varepsilon}D_{QW}\right)\\-\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW}+\mathrm{s}_{2\varepsilon}D_{QW}\right)\\\mathrm{c}_\varsigma Z_{QW}\end{matrix}\right\rangle\left\langle\begin{matrix}1\\-\mathrm{xs}_{2\varepsilon}\\\mathrm{xc}_{2\varepsilon}\\0\end{matrix}\right|$$

2 (S.10.18.17)

$$\frac{\left\langle \mathbf{A}_S(\mathrm{y}) \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon,\omega)\right)}{T_S T_{CP}} = \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon,\omega)\right)}{T_S T_{CP}} =$$

$$= \left\langle \begin{matrix} 1 \\ \mathrm{y}D_S \\ 0 \\ 0 \end{matrix} \middle\| \begin{matrix} 1 \\ \mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon} D_{QW}\right) \\ -\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} + \mathrm{s}_{2\varepsilon} D_{QW}\right) \\ \mathrm{c}_\omega Z_{QW} \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right| = \left[1+\mathrm{xy}D_S\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right)\right] \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right|$$

1 (S.10.18.18)

2 $$\frac{I_S}{\eta_S T_S T_{CP} T_O F_{11} T_E I_L} = \left[1+\mathrm{xy}D_S\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right)\right] \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \middle\| \begin{matrix} i_{in} \\ q_{in} \\ u_{in} \\ v_{in} \end{matrix} \right\rangle = \qquad \text{(S.10.18.19)}$$

$$= \left[1+\mathrm{xy}D_S\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right)\right]\left[i_{in} - \mathrm{x}\left(\mathrm{s}_{2\varepsilon}q_{in} - \mathrm{c}_{2\varepsilon}u_{in}\right)\right]$$

$$D_T = +1, D_R = -1 \Rightarrow$$

3 $$\frac{\eta^*}{\eta} = \frac{1-\mathrm{xy}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right)}{1+\mathrm{xy}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right)} \qquad \text{(S.10.18.20)}$$

$$D_Q = 0, Z_Q = 1 \Rightarrow$$

$$\frac{\left\langle \mathbf{A}_S(\mathrm{y}) \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon,\omega)\right)}{T_S T_{CP}} = \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon,\omega)\right)}{T_S T_{CP}} =$$

4 $$= \left\langle \begin{matrix} 1 \\ \mathrm{y}D_S \\ 0 \\ 0 \end{matrix} \middle\| \begin{matrix} 1 \\ \mathrm{xs}_{2\varepsilon}\mathrm{s}_\omega \\ -\mathrm{xc}_{2\varepsilon}\mathrm{s}_\omega \\ \mathrm{c}_\omega \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right| = \left[1+\mathrm{xy}D_S\mathrm{s}_{2\varepsilon}\mathrm{s}_\omega\right] \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right| \qquad \text{(S.10.18.21)}$$

$$\frac{\left\langle \mathbf{A}_S(\mathrm{y},\gamma) \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon)\right)}{T_S T_O T_{CP}} = \frac{\left\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{M}_O(\gamma) \middle| \mathbf{M}_{CP}(\mathrm{x}45^\circ+\varepsilon)\right)}{T_S T_O T_{CP}} =$$

5 $$= \left\langle \begin{matrix} 1+\mathrm{yc}_{2\gamma}D_S D_O \\ \mathrm{c}_{2\gamma}D_O + \mathrm{y}D_S\left(1-\mathrm{s}_{2\gamma}^2 W_O\right) \\ \mathrm{s}_{2\gamma}\left(D_O + \mathrm{yc}_{2\gamma}D_S W_O\right) \\ -\mathrm{ys}_{2\gamma}D_S Z_O \mathrm{s}_O \end{matrix} \middle\| \begin{matrix} 1 \\ \mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right) \\ -\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW} + \mathrm{s}_{2\varepsilon}D_{QW}\right) \\ \mathrm{c}_\varsigma Z_{QW} \end{matrix} \right\rangle \left\langle \begin{matrix} 1 \\ -\mathrm{xs}_{2\varepsilon} \\ \mathrm{xc}_{2\varepsilon} \\ 0 \end{matrix} \right| = \qquad \text{(S.10.18.22)}$$

$$= 1+\mathrm{y}D_S\left(\mathrm{c}_{2\gamma}D_O - \mathrm{s}_{2\gamma}Z_O\mathrm{s}_O\mathrm{c}_\varsigma Z_{QW}\right) +$$

$$+\mathrm{x}\left\{ \begin{matrix} \left[\mathrm{c}_{2\gamma}D_O + \mathrm{y}D_S\left(1-\mathrm{s}_{2\gamma}^2W_O\right)\right]\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW} - \mathrm{c}_{2\varepsilon}D_{QW}\right) - \\ -\mathrm{s}_{2\gamma}\left(D_O + \mathrm{yc}_{2\gamma}D_S W_O\right)\left(\mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW} + \mathrm{s}_{2\varepsilon}D_{QW}\right) \end{matrix} \right\}$$

$$\gamma = 0 \Rightarrow$$

$$\frac{\langle \mathbf{A}_S(\mathrm{y},0) | \mathbf{M}_{CP}(\mathrm{x}45° + \varepsilon)}{T_S T_O T_{CP}} = \frac{\langle \mathbf{M}_S \mathbf{R}_\mathrm{y} \mathbf{M}_O(0) | \mathbf{M}_{CP}(\mathrm{x}45° + \varepsilon)}{T_S T_O T_{CP}} = \\ = 1 + \mathrm{y}D_S D_O + \mathrm{x}\left\{\left[D_O + \mathrm{y}D_S\right]\left(\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma Z_{QW} - \mathrm{c}_{2\varepsilon} D_{QW}\right)\right\} \qquad \text{(S.10.18.23)}$$

CP with QWP without diattenuation: $D_{QW} = 0,\ Z_{QW} = 1,\ \Rightarrow$

$$\frac{\left[\mathbf{M}_{QW}\left(45°, \Delta_{QW}{=}90° + \varsigma\right)\mathbf{M}_P\right](\mathrm{x}45° + \varepsilon)}{T_{QW} T_P} = \\ = \begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon} D_P & . & . \\ \mathrm{xs}_{2\varepsilon}\mathrm{s}_\varsigma D_P & -\mathrm{s}_{2\varepsilon}^2 \mathrm{s}_\varsigma + \mathrm{c}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P + \mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{s}_P\right) Z_P & . & . \\ -\mathrm{xc}_{2\varepsilon}\mathrm{s}_\varsigma D_P & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma + \mathrm{c}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P - \mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{s}_P\right) Z_P & . & . \\ \mathrm{c}_\varsigma D_P & -\mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma + \mathrm{c}_{2\varepsilon}\mathrm{s}_\varsigma \mathrm{s}_P Z_P\right) & . & . \\ . & . & \mathrm{xc}_{2\varepsilon} D_P & 0 \\ . & . & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma + \mathrm{s}_{2\varepsilon}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_P + \mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{s}_P\right) Z_P & \mathrm{x}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{c}_P - \mathrm{c}_{2\varepsilon}\mathrm{s}_P\right) Z_P \\ . & . & -\mathrm{c}_{2\varepsilon}^2 \mathrm{s}_\varsigma + \mathrm{s}_{2\varepsilon}\left(\mathrm{s}_{2\varepsilon}\mathrm{c}_P - \mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{s}_P\right) Z_P & -\mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma \mathrm{c}_P + \mathrm{s}_{2\varepsilon}\mathrm{s}_P\right) Z_P \\ . & . & \mathrm{x}\left(\mathrm{c}_{2\varepsilon}\mathrm{c}_\varsigma - \mathrm{s}_{2\varepsilon}\mathrm{s}_\varsigma \mathrm{s}_P Z_P\right) & -\mathrm{s}_\varsigma \mathrm{c}_P Z_P \end{pmatrix} \qquad \text{(S.10.18.24)}$$

CP with QW without diattenuation and phase shift error: $D_{QW} = 0,\ Z_{QW} = 1,\ \varsigma = 0 \Rightarrow$

$$\frac{\left[\mathbf{M}_{QW}\left(45°, \Delta_{QW}{=}90°\right)\mathbf{M}_P\right](\mathrm{x}45° + \varepsilon)}{T_{QW} T_P} = \begin{pmatrix} 1 & -\mathrm{xs}_{2\varepsilon} D_P & \mathrm{xc}_{2\varepsilon} D_P & 0 \\ 0 & \mathrm{c}_{2\varepsilon}\mathrm{c}_{2\varepsilon-P} Z_P & \mathrm{s}_{2\varepsilon}\mathrm{c}_{2\varepsilon-P} Z_P & \mathrm{xs}_{2\varepsilon-P} Z_P \\ 0 & \mathrm{c}_{2\varepsilon}\mathrm{s}_{2\varepsilon-P} Z_P & \mathrm{s}_{2\varepsilon}\mathrm{s}_{2\varepsilon-P} Z_P & -\mathrm{xc}_{2\varepsilon-P} Z_P \\ D_P & -\mathrm{xs}_{2\varepsilon} & \mathrm{xc}_{2\varepsilon} & 0 \end{pmatrix}$$

(S.10.18.25)

S.10.19 Circular analyser (CA)

(see Chipman (2009a) Chap. 15.26)

In order to keep the same flexibility regarding the mutual orientation between the linear polariser and the λ/4 plate as for the circular polariser, we construct the ideal circular analyser with a λ/4 plate at ±45° and a linear polariser at 0° or 90° (according to Chipman (2009a) 15.18: left circular analyser for x,z = +1) from Eqs. (S.10.8.5) and (S.10.17.4)

$$
\frac{\mathbf{M}_{CA}\left(\mathrm{x},\mathrm{z}\right)}{T_{CA}}=\frac{\mathbf{M}_{P}\left(45°-\mathrm{x}45°\right)}{T_{P}}\frac{\mathbf{M}_{QW}\left(z45°\right)}{T_{QW}}=
$$

$$
=\left|\begin{matrix}1\\ \mathrm{x}\\ 0\\ 0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\ \mathrm{x}\\ 0\\ 0\end{matrix}\right|\begin{pmatrix}1&0&0&0\\ 0&0&0&-\mathrm{z}\\ 0&0&1&0\\ 0&\mathrm{z}&0&0\end{pmatrix}=\left|\begin{matrix}1\\ \mathrm{x}\\ 0\\ 0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\ 0\\ 0\\ -\mathrm{xz}\end{matrix}\right|=\begin{pmatrix}1&0&0&-\mathrm{zx}\\ \mathrm{x}&0&0&-\mathrm{z}\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix} \qquad \text{(S.10.19.1)}
$$

with $\mathrm{x},\mathrm{z}=\pm1$

with $\mathrm{zx}=+1\Rightarrow$ left circ. analyzer

with $\mathrm{zx}=-1\Rightarrow$ right circ. analyzer

Ideal circular analyser with QWP at z45° with error angle ε

$$
\frac{\mathbf{M}_{CA}\left(\mathrm{x},\mathrm{z},\varepsilon\right)}{T_{CA}}=\mathbf{R}\left(+\varepsilon\right)\frac{\mathbf{M}_{CA}\left(\mathrm{x},\mathrm{z}\right)}{T_{CA}}\mathbf{R}\left(-\varepsilon\right)=
$$

$$
=\begin{pmatrix}1&0&0&0\\ 0&\mathrm{c}_{2\varepsilon}&-\mathrm{xs}_{2\varepsilon}&0\\ 0&\mathrm{xs}_{2\varepsilon}&\mathrm{c}_{2\varepsilon}&0\\ 0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&-\mathrm{zx}\\ \mathrm{x}&0&0&-\mathrm{z}\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\ 0&\mathrm{c}_{2\varepsilon}&\mathrm{xs}_{2\varepsilon}&0\\ 0&-\mathrm{xs}_{2\varepsilon}&\mathrm{c}_{2\varepsilon}&0\\ 0&0&0&1\end{pmatrix}= \qquad \text{(S.10.19.2)}
$$

$$
=\begin{pmatrix}1&0&0&-\mathrm{zx}\\ \mathrm{xc}_{2\varepsilon}&0&0&-\mathrm{zc}_{2\varepsilon}\\ \mathrm{s}_{2\varepsilon}&0&0&-\mathrm{zxs}_{2\varepsilon}\\ 0&0&0&1\end{pmatrix}=\left|\begin{matrix}1\\ \mathrm{xc}_{2\varepsilon}\\ \mathrm{s}_{2\varepsilon}\\ 0\end{matrix}\right\rangle\left\langle\begin{matrix}1\\ 0\\ 0\\ -\mathrm{zx}\end{matrix}\right|
$$

S.11 Helpful relations

(see also S.10.10)

$$\frac{1-\delta}{1+\delta} = a = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}}, \quad \delta = \frac{1-a}{1+a} \tag{S.11.1}$$

$$D_T \equiv \frac{T_T^p - T_T^s}{T_T^p + T_T^s}, \quad T_T \equiv 0.5\left(T_T^p + T_T^s\right) \tag{S.11.2}$$

$$\left.\begin{array}{l} 1 - D_T = 1 - \dfrac{T_T^p - T_T^s}{T_T^p + T_T^s} = \dfrac{2T_T^s}{T_T^p + T_T^s} = \dfrac{T_T^s}{T_T} \\ 1 + D_T = 1 - \dfrac{T_T^p - T_T^s}{T_T^p + T_T^s} = \dfrac{2T_T^p}{T_T^p + T_T^s} = \dfrac{T_T^p}{T_T} \end{array}\right\} \Rightarrow \frac{1+D_T}{1-D_T} = \frac{T_T^p}{T_T^s} \tag{S.11.3}$$

$$\begin{aligned} 1 + D_O D_T &= 1 + \frac{T_O^p - T_O^s}{T_O^p + T_O^s}\frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{\left(T_O^p + T_O^s\right)\left(T_T^p + T_T^s\right) + \left(T_O^p - T_O^s\right)\left(T_T^p - T_T^s\right)}{\left(T_O^p + T_O^s\right)\left(T_T^p + T_T^s\right)} = \\ &= \frac{T_O^p T_T^p + T_O^s T_T^s}{2T_O T_T} \\ 1 - D_O D_T &= 1 - \frac{T_O^p - T_O^s}{T_O^p + T_O^s}\frac{T_T^p - T_T^s}{T_T^p + T_T^s} = \frac{\left(T_O^p + T_O^s\right)\left(T_T^p + T_T^s\right) - \left(T_O^p - T_O^s\right)\left(T_T^p - T_T^s\right)}{\left(T_O^p + T_O^s\right)\left(T_T^p + T_T^s\right)} = \\ &= \frac{T_O^s T_T^p + T_O^p T_T^s}{2T_O T_T} \end{aligned} \tag{S.11.4}$$

$$D_O = 0 \Rightarrow Z_O = \sqrt{1 - D_O^{\;2}} = 1, \quad W_O = 1 - \mathrm{c}_O \tag{S.11.5}$$

$$\left|D_O\right| = 1 \Rightarrow Z_O = \sqrt{1 - D_O^{\;2}} = 0, \quad W_O = 1 \tag{S.11.6}$$

1 **S.12 Trigonometric relations**

2
$$\left.\begin{aligned} s_\phi c_\phi &= \tfrac{1}{2} s_{2\phi} \\ c_\alpha c_\beta &= \tfrac{1}{2}\left(c_{\alpha-\beta} + c_{\alpha+\beta}\right) \\ s_\alpha s_\beta &= \tfrac{1}{2}\left(c_{\alpha-\beta} - c_{\alpha+\beta}\right) \end{aligned}\right\} \Rightarrow \begin{cases} c_\phi c_\beta + s_\phi s_\beta = \tfrac{1}{2}\left(c_{\phi-\beta} + c_{\phi+\beta} + c_{\phi-\beta} - c_{\phi+\beta}\right) = c_{\phi-\beta} \\ c_\phi c_\beta - s_\phi s_\beta = \tfrac{1}{2}\left(c_{\phi-\beta} + c_{\phi+\beta} - c_{\phi-\beta} + c_{\phi+\beta}\right) = c_{\phi+\beta} \\ s_\phi c_\beta - c_\phi s_\beta = \tfrac{1}{2}\left(s_{\phi-\beta} + s_{\phi+\beta} - s_{\phi+\beta} + s_{\phi-\beta}\right) = s_{\phi-\beta} \\ s_\phi c_\beta + c_\phi s_\beta = \tfrac{1}{2}\left(s_{\phi-\beta} + s_{\phi+\beta} + s_{\phi+\beta} - s_{\phi-\beta}\right) = s_{\phi+\beta} \end{cases} \tag{S.12.1}$$

3
$$\begin{aligned} &\text{with } \phi = x45° + \varepsilon, x = \pm 1 \Rightarrow \\ &c_{2\phi} = \cos\left[2\left(x45° + \varepsilon\right)\right] = \cos(\pm 90° + 2\varepsilon) = \mp\sin(2\varepsilon) = -xs_{2\varepsilon} \\ &s_{2\phi} = \sin\left[2\left(x45° + \varepsilon\right)\right] = \sin(\pm 90° + 2\varepsilon) = \pm\cos(2\varepsilon) = xc_{2\varepsilon} \\ &\text{with } \phi = x45° + 45 + \varepsilon, x = \pm 1 \Rightarrow \\ &c_{2\phi} = \cos\left[2\left(x45° + 45 + \varepsilon\right)\right] = \cos(\pm 90° + 90° + 2\varepsilon) = \mp\cos(2\varepsilon) = -xc_{2\varepsilon} \\ &s_{2\phi} = \sin\left[2\left(x45° + 45 + \varepsilon\right)\right] = \sin(\pm 90° + 90° + 2\varepsilon) = \mp\sin(2\varepsilon) = -xs_{2\varepsilon} \end{aligned} \tag{S.12.2}$$

$$\textit{for } \left(x45° + \varepsilon\right) \to \left(x45° + 45 + \varepsilon\right) \Rightarrow \begin{cases} -xs_{2\varepsilon} \to -xc_{2\varepsilon} \\ xc_{2\varepsilon} \to -xs_{2\varepsilon} \end{cases}$$

4
$$\begin{aligned} &\text{with } x = \pm 1 \Rightarrow \\ &\cos\left[2\left(x45°\right)\right] = 0 \\ &\sin\left[2\left(x45°\right)\right] = x \\ &\cos\left[2\left(x45° + 45\right)\right] = -x \\ &\sin\left[2\left(x45° + 45\right)\right] = 0 \\ &\cos\left[2\left(-x45° - \gamma\right)\right] = \cos\left(\mp 90° - 2\gamma\right) = \mp\sin\left(2\gamma\right) = -xs_{2\gamma} \\ &\sin\left[2\left(-x45° - \gamma\right)\right] = \sin\left(\mp 90° - 2\gamma\right) = \mp\cos\left(2\gamma\right) = -xc_{2\gamma} \\ &\cos\left\{2\left[x\left(45° + \gamma\right)\right]\right\} = \cos\left[\pm\left(90° + 2\gamma\right)\right] = -\sin\left(2\gamma\right) = -s_{2\gamma} \\ &\sin\left\{2\left[x\left(45° + \gamma\right)\right]\right\} = \sin\left[\pm\left(90° + 2\gamma\right)\right] = \pm\cos\left(2\gamma\right) = xc_{2\gamma} \end{aligned} \tag{S.12.3}$$

5
$$\begin{aligned} &\phi = x22.5° + \varepsilon/2, x = \pm 1 \Rightarrow \\ &c_{4\phi} = \cos\left[4\left(x22.5° + \varepsilon/2\right)\right] = \cos\left(x90° + 2\varepsilon\right) = -xs_{2\varepsilon} \\ &s_{4\phi} = \sin\left[4\left(x22.5° + \varepsilon/2\right)\right] = \sin\left(x90° + 2\varepsilon\right) = xc_{2\varepsilon} \end{aligned} \tag{S.12.4}$$

6
$$\begin{aligned} &s_{4\phi} = 2s_{2\phi}c_{2\phi} \\ &c_{4\phi} = c_{2\phi}^2 - s_{2\phi}^2 = 2c_{2\phi}^2 - 1 = 1 - 2s_{2\phi}^2 \Rightarrow \begin{cases} 1 + c_{4\phi} = 2c_{2\phi}^2 \\ 1 - c_{4\phi} = 2s_{2\phi}^2 \end{cases} \end{aligned} \tag{S.12.5}$$

$$
\begin{aligned}
&1-\mathrm{c}_{2\varepsilon}^{2}W_P=1-\left(1-\mathrm{s}_{2\varepsilon}^{2}\right)W_P=1-W_P+\mathrm{s}_{2\varepsilon}^{2}W_P=Z_P\mathrm{c}_P+\mathrm{s}_{2\varepsilon}^{2}W_P=Z_P\mathrm{c}_P+\mathrm{s}_{2\varepsilon}^{2}\left(1-Z_P\mathrm{c}_P\right)\Rightarrow\\
&1-\mathrm{c}_{2\varepsilon}^{2}W_P=\mathrm{s}_{2\varepsilon}^{2}+\mathrm{c}_{2\varepsilon}^{2}Z_P\mathrm{c}_P\\
&1-\mathrm{s}_{2\varepsilon}^{2}W_P=\mathrm{c}_{2\varepsilon}^{2}+\mathrm{s}_{2\varepsilon}^{2}Z_P\mathrm{c}_P
\end{aligned}
\qquad \text{(S.12.6)}
$$

S.12.1 Tangent half-angle substitution

The substitution Eq. (S.12.1.1) is sometimes called Weierstrass substitution, but it can already be found in Euler's Institutionum calculi integralis (Eneström number E342: Vol. 1 Part 1, Sect. 1, Chap. 5, Problem 29, http://eulerarchive.maa.org/pages/E342.html).

$$
t=\tan\left(\frac{\theta}{2}\right)\Leftrightarrow\sin\theta=\frac{2t}{1+t^{2}} \qquad \text{(S.12.1.1)}
$$

With this substitution we can write Eq. (S.12.1.2) and yield ε from Eq. (S.12.1.3). For small ε we get the approximation Eq. (S.12.1.4).

$$
t=K\mathrm{s}_{2\varepsilon}=\tan\left(\frac{\theta}{2}\right)\Leftrightarrow\sin\theta=Y=\frac{2K\mathrm{s}_{2\varepsilon}}{1+\left(K\mathrm{s}_{2\varepsilon}\right)^{2}} \qquad \text{(S.12.1.2)}
$$

$$
\varepsilon=\frac{1}{2}\arcsin\left[\frac{1}{K}\tan\left(\frac{\arcsin\left(Y\right)}{2}\right)\right] \qquad \text{(S.12.1.3)}
$$

$$
\begin{aligned}
&K<1\wedge\varepsilon\ll1\Rightarrow\\
&\varepsilon\approx\frac{Y}{2K}
\end{aligned}
\qquad \text{(S.12.1.4)}
$$

S.13 Example

From Eqs. (65) and (61)

$$F_{11} \propto \eta H_R I_T - H_T I_R \quad \text{and} \quad \delta = \frac{\delta^* \left(G_T + H_T\right) - \left(G_R + H_R\right)}{\left(G_R - H_R\right) - \delta^* \left(G_T - H_T\right)}$$

and the relations

$$D_R = \frac{T_R^p - T_R^s}{T_R^p + T_R^s},\ D_T = \frac{T_T^p - T_T^s}{T_T^p + T_T^s},\ T_R = \frac{T_R^p + T_R^s}{2},\ T_T = \frac{T_T^p + T_T^s}{2},\ T^* = \frac{T_T}{T_R} \qquad \text{(S.13.1)}$$
$$1 - D_R = T_R^s / T_R,\ 1 + D_R = T_R^p / T_R,\ 1 - D_T = T_T^s / T_T,\ 1 + D_T = T_T^p / T_T$$

we get for the simplest case from Eq. (79):

90° rotated polarising beam-splitter

$$\text{with } \mathrm{y} = -1 \Rightarrow \Psi = 90°;\ G_S = 1,\ H_S = -D_S \Rightarrow$$
$$\delta = \frac{\delta^* T_T^s - T^* T_R^s}{T^* T_R^p - \delta^* T_T^p} \qquad \text{(S.13.2)}$$

$$F_{11} \propto -\eta D_R I_T + D_T I_R = \frac{T_T^p - T_T^s}{T_T^p + T_T^s} I_R - \frac{T_R^p - T_R^s}{T_R^p + T_R^s} \eta I_T \qquad \text{(S.13.3)}$$

0° rotated polarising beam-splitter

$$\text{with } \mathrm{y} = +1 \Rightarrow \Psi = 0°;\ G_S = 1,\ H_S = D_S \Rightarrow$$
$$\delta = \frac{\delta^* T_T^p - T^* T_R^p}{T^* T_R^s - \delta^* T_T^s} \qquad \text{(S.13.4)}$$

$$F_{11} \propto \eta D_R I_T - D_T I_R = -\frac{T_T^p - T_T^s}{T_T^p + T_T^s} I_R + \frac{T_R^p - T_R^s}{T_R^p + T_R^s} \eta I_T \qquad \text{(S.13.5)}$$

From Eq. (78) with cleaned 90° rotated polarising beam-splitter =>

$$\text{with } D_T = 1,\quad D_R = -1 \Rightarrow$$
$$G_T = 1 + \mathrm{y} D_O,\quad G_R = 1 - \mathrm{y} D_O,$$
$$H_T = D_O + \mathrm{y} = \mathrm{y} G_T,\quad H_R = D_O - \mathrm{y} = -\mathrm{y} G_R \qquad \text{(S.13.6)}$$
$$T_O = \frac{T_O^p + T_O^s}{2},\quad 1 - D_O = T_O^s / T_O,\ 1 + D_O = T_O^p / T_O$$

$$\delta = \frac{\delta^*(1+y) - \frac{1-yD_O}{1+yD_O}(1-y)}{\frac{1-yD_O}{1+yD_O}(1+y) - \delta^*(1-y)}$$

$$\text{with } y=-1 \Rightarrow \delta = \frac{1}{\delta^*}\frac{1+D_O}{1-D_O} = \frac{1}{\delta^*}\frac{T_O^p}{T_O^s} \tag{S.13.7}$$

$$\text{with } y=+1 \Rightarrow \delta = \delta^*\frac{1+D_O}{1-D_O} = \delta^*\frac{T_O^p}{T_O^s}$$

$$F_{11} \propto \eta(D_O - y)I_T - (D_O + y)I_R$$

$$\text{with } y=-1 \Rightarrow F_{11} \propto I_T + \frac{1}{\eta}\frac{1-D_O}{1+D_O}I_R = I_T + \frac{1}{\eta}\frac{T_O^s}{T_O^p}I_R \tag{S.13.8}$$

$$\text{with } y=+1 \Rightarrow F_{11} \propto I_T + \frac{1}{\eta}\frac{1+D_O}{1-D_O}I_R = I_T + \frac{1}{\eta}\frac{T_O^p}{T_O^s}I_R$$

S.14 Determination of the degree of circular polarisation of the emitted laser beam.

Combining the ±45 calibrations with different calibrators can yield information about the laser polarisation. With a cleaned analyser, without receiver optics rotation, and with a QWP without calibrator rotation ε before the receiving optics from Chap. (9.2) and a linear polariser (LP) from Chap. (8.2)

$$\text{with } \gamma = \varepsilon = 0, D_T = +1, D_R = -1 \Rightarrow$$

$$\frac{\eta^*_{QWP}}{\eta} = \frac{1-yD_O}{1+yD_O}\frac{i_E + xy(1-2a)v_E}{i_E - xy(1-2a)v_E} \tag{S.14.1}$$

$$\frac{\eta^*_{LP,\Delta 90}}{\eta} = \frac{1-yD_O}{1+yD_O}$$

we can calculate

$$Y \equiv \frac{\eta^*_{QWP}(x)}{\eta^*_{LP,\Delta 90}} = \frac{i_E + xy(1-2a)v_E}{i_E - xy(1-2a)v_E} \tag{S.14.2}$$

and get the degree of circular polarisation

$$\frac{v_E}{i_E} = \frac{1}{xy(1-2a)}\frac{(Y-1)}{(Y+1)} \tag{S.14.3}$$