

Truth and Uncertainty. A critical discussion of the error concept versus the uncertainty concept

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Abstract. Contrary to the statements put forward in “Evaluation of measurement data – Guide to the expression of uncertainty in measurement” ([GUM08, edition 2008 \(GUM-2008\)](#)), issued by the Joint Committee for Guides in Metrology, the error concept and the uncertainty concept are the same. Arguments in favour of the contrary have been analyzed and were found not compelling. Neither was any evidence presented in [GUM08-GUM-2008](#) that “errors” and “uncertainties” define a different relation between the measured and the true value of the variable of interest, nor does this document refer to a Bayesian account of uncertainty beyond the mere endorsement of a degree-of-belief-type conception of probability.

1 Introduction

~~For more than 200 years, error estimation used a more or less unified terminology where the term ‘error’ was used, with some caveats, for designating a statistical estimate of the expected difference between the measured and the true value of a measurand (Gauss, 1809; Gauss, 1816; Pearson, 1920; Fisher, 1925; Rodgers, 1990; Mayo, 1996; Rodgers, 2000; just to name a few). It has long been recognized that the quantitative characterization of the reliability of a measurement is essential to draw quantitative conclusions from the measured data. Various and often contradicting methods and terminologies emerged over the years. The activity ‘Towards Unified Error Reporting, (TUNER),’ aims at a unification of the reporting of errors in estimates of atmospheric state variables retrieved from satellite measurements (von Clarmann et al., 2020)). On request of the Bureau International de Poids et Mesures (BIPM) presented a contrasting definition how we have to conceive the term ‘error’ and have stipulated a new terminology, where the term ‘measurement uncertainty’ is used in situations where one would have said ‘measurement error’ before (Joint Committee for Guides in Metrology (JCGM), 2008, this source is henceforth referenced as GUM08), the Joint Committee for Guides in Metrology (JCGM) has issued a guideline how measurement uncertainty should be dealt with~~

25 (Joint Committee for Guides in Metrology (JCGM), 2008, this source is henceforth referenced as GUM-2008).
Supplementary material in the context of GUM is found in Joint Committee for Guides in Metrology
(JCGM) (2012) and several supplements to ~~GUM08~~GUM-2008, that are found on the BIPM website
(<https://www.bipm.org/en/publications/guides/gum.html>). The ~~new concept~~concept of uncertainty
was developed long before the GUM-2008 was issued and has seen several refinements since then
30 (e.g. Eisenhart and Collé, 1980; Collé, 1987; Colclough, 1987; Schumacher, 1987). A key claim
of GUM-2008 is that the terms “error” and “uncertainty” connote different things, and that the
underlying concepts are different. GUM-2008 has been critically discussed by, e.g., Bich (2012),
Grégis (2015), Elster et al. (2013) and The European Centre for Mathematics and Statistics in
Metrology (2019), and more favorably by, e.g., Kacker et al. (2007). ~~The claim is made that the~~
35 ~~uncertainty concept can be construed without reference to the unknown and unknowable true~~
~~value while the error concept cannot (GUM08, p.3 and p. 5). Thus, the dispute between the error~~
~~statisticians and the uncertainty statisticians comes down to the question if and how the error (or~~
~~uncertainty) distribution is related to the true value of the measurand.~~

In this paper we ~~try to shed some light upon this relation which seems to have caused a rift~~
40 ~~both in the communities of statistics and~~critically asses some of empirical sciences. Further, we
~~critically discuss the applicability of the GUM08 recommendations in the context of~~the claims
made in GUM-2008 and, as part of the TUNER activity, discuss its applicability to remote sensing
of the atmosphere. ~~Remote sounding employs indirect measurements where the measurand is not~~
~~measured directly but retrieved from the measured signal by the inverse solution of the radiative~~
45 ~~transfer equation which provides the link between the measurand and the measured signal. In the~~
~~context of the work undertaken by the activity ‘Towards Unified Error Reporting, (TUNER),’ a~~
~~project aiming at unification of error reporting of satellite data (von Clarmann et al., 2020)), this~~
~~issue is particularly problematic. Without agreement on the concepts and the terminology of error~~
~~versus uncertainty assessment, any unification is out of reach.~~

50 ~~At the outset we recapitulate the concept of indirect measurements and lay down an appropriate~~
~~terminology and notation (Section??). In the subsequent section (Section ??), we analyze the use of~~
~~the term ‘error’ by the uncertainty statisticians¹ and will find that it is often not consistent with~~
~~the use of this term as originally used by the error statisticians. Then we try to find out what~~
~~the exact connotation of the term ‘uncertainty’ is and how it is actually distinguished from the~~
55 ~~traditional concept of error analysis (Section ??). We start with analyzing GUM’s claim about the~~
differences between error and uncertainty (Section 2), whereby it is important to distinguish between

¹We use the term ‘uncertainty concept’ for a concept where it is claimed that error and uncertainty are different entities and that ‘uncertainty’ can be defined without reference to the true value of the measurand. We use the term ‘uncertainty scientists’ or ‘uncertainty statisticians’ for scientists endorsing the uncertainty analysis concept. Conversely, we use the term ‘error analysis concept’ for a concept which denies a fundamental difference between the traditional concept of error analysis and the uncertainty concept as endorsed by GUM08², and we call ‘error scientists’ or ‘error statisticians’ those scientists who endorse the traditional concept of error analysis.

terminological (Section 2.1) and conceptual (Section 2.2) issues. We shall find that the concept of the ‘true value of the measurand’ makes up the alleged key difference. That is to say, the uncertainty concept endorsed by GUM is claimed, contrary to traditional error analysis, to be able to dispense
60 with the concept of the true value ~~-.The problem of the true value is that it that is neither known nor knowable -.In Section 3 we discuss how this affects error and uncertainty estimation and the relation between the measured and (GUM-2008, p.3 and p. 5).~~ This leads to the question of whether, and if so, how the measured or estimated value along with the estimated error (or uncertainty) are related to the true value ~~-.We find (1) that according to Bayesian statistics (Bayes, 1763) the measured value cannot always be interpreted as the most probable value of the measurand -. (2) We further find that nonlinear relationships between the measurand and the measured signal poses problems to measurand has in reality and what the problems related to the ignorance of the uncertainty analysis because a value in sufficient proximity to the true value should be chosen as linearization point for uncertainty estimation and thus must be — at least approximately — known; and (3) we accept~~
65 that we can never know for certain if the error or uncertainty budget is complete. In the following we investigate the implications of these three problems in turn. First are in the context of error estimation (Section 3). In this context, we first address the question, if it is adequate to use the true value, which is typically unknown and unknowable, in the definition of the term error, and to base error analysis on such a definition (Section 3.1). Second, we investigate under which conditions an error or uncertainty distribution can be understood as a distribution which tells us which likelihood or probability we can assign to a value to be the implications that the inverse nature of a measurement process has on the probabilistic relationship between the measured value, the true value, given a certain measured value (Section ??). We shall see that the interpretation of resulting error or uncertainty estimates is completely different in a maximum likelihood versus a Bayesian framework. ~~Second we assess to which degree and the measurement (Section 3.2. In this context we discuss the problem of the baserate fallacy. Further, it is investigated if the alleged difference between the error concept and the uncertainty concept can be explained by a Bayesian turn in metrology. Third, we assess the degree to which~~ the nonlinearity of the relationship between the measured signal and the target quantity, *viz.*, the radiative transfer equation, poses additional problems (Section 3.3). And ~~third-fourth,~~ we scruti-
70 nize the claim that there will always be unknown sources of uncertainty and that it is thus impossible to relate the measured value along with its uncertainty estimate to the true value (Section 3.4). After these more general considerations we critically discuss the applicability of the ~~GUM08-GUM-2008~~ concept to indirect measurements of atmospheric state variables (Section 4). There we discuss the problems of measurands that are not well-defined in the sense of ~~GUM08-GUM-2008~~ (Section 4.1),
75 and if it is really adequate to report the combined error only (Section 4.2)~~and if the measurement equation as presented in GUM08 does optimally support the uncertainty assessment in atmospheric remote sensing (Section ??) In this context, we also investigate whether the difference between the traditional concept of error analysis and the uncertainty concept might be linked to a Bayesian versus~~

a frequentist conception of probability. Finally (Section 5) we conclude to which degree the degree

95 to which the arguments put forward by the Joint Committee for Guides in Metrology (JCGM) are conclusive and what the differences between the error concept and the uncertainty concept actually are.

2 Recapitulation of the concept of indirect measurements Error and Uncertainty

In the case of indirect measurements, e.g., remote sensing, the measurand, i.e., the quantity of
100 interest, x , and the measured signal y are linked via a function f as

$$y = f(x; b) + \epsilon,$$

where b represents the parameters of f representing physical side conditions and ϵ is the actual measurement error in the y -domain (Rodgers, 2000) GUM-2008 endorses a new terminology compared to that of traditional error analysis. In the case of remote sensing of the atmosphere
105 f is the radiative transfer function. We use vector notation because in remote sensing typically multiple measurands are estimated from multiple measurement signals. For example, y could be a spectral measurement of an ozone emission line in the infrared; x could be a vertical profile of ozone concentrations, and b could include a vertical profile of temperature, known a priori with some uncertainty and affecting the signal in the ozone line. To obtain information on the measurand,
110 some kind of inverse solution of Equation ?? is required, because the estimate of the target quantity x involves a conclusion from the effect to the cause. More often than not, this inverse problem is ill-posed in the sense of Hadamard (1902), and the direct inversion is impossible and some kind of workaround is employed. Candidate workarounds are least-squares solutions, regularized solutions and so forth. von Clarmann et al. (2020) summarize the most common methods to solve this kind of
115 problem, along with related error estimation schemes. Here we call this substitute for the genuine inversion \tilde{F}^{-1} . Here F is a function representing the true radiative transfer function to the best of our knowledge, i. e., the descriptive radiative transfer law as opposed to the governing but unknown law. The $\tilde{\cdot}$ symbol reminds us that the inversion is not necessarily a genuine inversion in a mathematical sense. With this an estimate of the measurand can be obtained¹:

$$120 \hat{x} = \tilde{F}^{-1}(y; b)$$

Differences between the estimate¹ \hat{x} and the true value of the measurand x can be due to measurement errors represented by ϵ , erroneous assumptions on the values of the parameter vector

¹Strictly speaking, it would be adequate to assign a different symbol to the vector of parameters b when it appears in the context of f , where it designates the true parameters, and in the context of F , where it represents estimates or uncertain a priori knowledge about these parameters. However, we assume that it is clear from the context what is meant.

¹In the context of direct measurements, 'estimated value' and 'measured value' connote the same thing; in the context of indirect measurements, we use the term 'estimated value' for the value of the target quantity that was inferred from the measured signal.

~~b , context of the work undertaken by the TUNER activity, a project aiming at unification of error reporting of satellite data (von Clarmann et al., 2020)], terminological and conceptual divergence is particularly problematic. Without agreement on the concepts and the terminology of error versus uncertainty assessment, any unification is out of reach.~~

According to GUM-2008, the concept of uncertainty analysis should replace the concept of error analysis. The International Vocabulary of Metrology document (Joint Committee for Guides in Metrology (JCGM), 2012) points in the same direction. Thus, some conceptual and terminological remarks seem appropriate. While, on the face of it, this is quibbling about words, actually conceptual differences between the radiative transfer model F and the true but not exactly known radiative transfer physics f , and characteristics of \tilde{F}^{-1} , i.e., tricks applied to make a non-invertible inverse problem solvable. These error sources and recipes to estimate related error components of \hat{x} are discussed in depth in von Clarmann et al. (2020), drawing upon Rodgers (2000) and Rodgers (1990) errors and uncertainties are claimed to exist. This issue is discussed in the following.

Some complication arises because the true atmospheric state can be represented only by spatially continuous functions while we work with vectors of finite dimension. Here we avoid related difficulties by assuming that the measurand x represents a discretized atmosphere, i.e., it does not represent a point value but some kind of spatio-temporal average.

All this holds A key claim of GUM-2008 is that the “concept of *mutatis mutandis* uncertainty also for scalar quantities where

$$y = f(x) + \epsilon$$

but this distinction has no bearing upon our argument.

When reading the thermometer, we actually read the length of the mercury column, apply inversely the law of thermal expansion, and get an estimate of the temperature:-

$$\hat{x} = \tilde{F}^{-1}(y; b)$$

Only in trivial cases, when a measurement device has a calibrated scale from which \hat{x} can be directly read, F is unity. Here the inverse process is effectively pretabulated in the scale. In any case, the availability of \tilde{F}^{-1} allows to statistically estimate the effect of measurement noise ϵ and systematic effects in b or \tilde{F}^{-1} on \hat{x} (Rodgers, 1990, 2000; von Clarmann et al., 2020) as a quantifiable attribute is relatively new in the history of measurements, although *error* and *error analysis* have long been part of the practice of measurement science or metrology” (Joint Committee for Guides in Metrology (JCGM), 2008, Sect 0.2, l. 1–2; emphases in the original.). In a note to their Section 3.2.3, GUM-2008 states that “The terms ‘error’ and ‘uncertainty’ should be used properly and care taken to distinguish between them.” The discussion of these issues is occasionally led astray, because it is often not distinguished between two different questions: first,

whether the terms ‘error’ and ‘uncertainty’ have the same connotation, and second, whether the underlying concepts are indeed different. In the following, we try to shed some light on these issues.

160 3 **The connotation of the term ‘error’**

2.1 Terminological Issues

Although GUM08 (Sect. 0.2) claims the “concept of uncertainty as a quantifiable attribute to be a relatively new concept in the history of measurement”, we uphold the view that it has long been recognized that the result of a measurement remains to some degree uncertain even
165 when a thorough measurement procedure and error evaluation is performed. We recall that even ancient researchers realized that measurement results always have errors. Carl Friedrich (Gauss, 1809) and Adrien-Marie Legendre (1805) formalized the required procedure of balancing imperfect astrometric measurements by least squares fitting, in support of orbital calculations from overdetermined data sets. And there is no reason to believe that earlier investigators were unaware
170 of the fact that they were not working on perfect observational data. Kepler’s conclusion concerning the elliptical shape of Already in the pre-GUM language there have been subtle linguistic differences between the terms ‘error’ and ‘uncertainty’. The error has been conceived as an attribute of a measurement or an estimate, while the term ‘uncertainty’ has been used as an attribute of the true state. We perfectly know our measurement – even if it is erroneous – and thus we are uncertain about the true value. Because of the measurement error there is an uncertainty as to what the true value is. The uncertainty thus describes the degree of ignorance about the true value while the estimated error describes the degree to which the measurement is thought to deviate from the true value. In this use of language, both terms still relate to the same concept. This notion seems, as far as we can judge, to be consistent with the language widely used in the orbit of Mars based on the rich observational dataset collected by Brahe would have been impossible without proper implicit assumptions concerning the limited validity of the reported values (Kepler, 1609).

A rich methodical toolbox for error estimation and uncertainty assessment has been developed since then, and a confusing plethora of conventions how to communicate measurement uncertainties exists. Unification of error or uncertainty reporting is overdue but this requires, at a minimum,
185 agreement on terminology and the underlying concepts. While many of the technical terms used are quite clear and often self-explanatory, there is a particularly troublesome terminological issue. It is related to the use of the term ‘error’ and the underlying concept. According to the Joint Committee for Guides in Metrology (JCGM), the concept of uncertainty analysis should replace the concept of error analysis pre-GUM literature since Gauss (1809), who used the latin terms *error* and *incertitudo* in this way. Thus, some conceptual and terminological remarks seem appropriate. While on the face of it, this is quibbling about words, actually conceptual differences between the errors and uncertainties are claimed to exist. This issue is discussed in the following. both terms have referred

to the same thing but from a different perspective¹. The estimate of the total error includes both measurement noise and all known components of further errors, random or systematic, caused by imperfections in the measurement and data analysis system.

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~~In the context of measurements, It must, however, be noted, that the term ‘error’ traditionally has two slightly different connotations. The first is the is an equivocation. It has been used both for the unknown and unknowable signed actual difference between the measured or, in the context of indirect measurements, retrieved, value and the true value of the measurand; the second meaning of the term ‘error’ is, and for a statistical estimate of this difference. Often some attributes are used for clarification and specification, e.g., ‘probable error’ (Gauss, 1816; Bich, 2012), ‘statistical error’ Nuzzo (2014), ‘error estimation’ (Zhang et al., 2010) or ‘error analysis’ (Rodgers, 1990, 2000; Hughes and Hase, 2010). GUM08 (Annex B.2.19) allows only the connotation ‘actual difference’, but their use of the terms ‘error and error analysis’ in the first sentence of their 0.2 or ‘possible error’ in their 2.2.4 only make sense if the statistical meaning of the term ‘error’ is conceded. The authors of this paper are not aware of any case where the ambiguity of these connotations of the term ‘error’ has ever led to any misunderstanding, most probably because it is always clear from the context what is meant.~~

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~~In the case of ‘error’, its statistical estimate is mostly understood to be a quadratic estimate and thus does it. The statistical estimate is mostly understood to be the square root of the variance of the probability density function of the error²³ and thus does not carry any information about the sign of the error. Nonlinear error propagation may in some cases make asymmetric error estimates necessary, but typically these do not carry any information about the on the actual sign of the error. This either. The ignorance of the sign of the error entails that the true or most probable value cannot simply be determined by subtracting the estimated error from the measured value.~~

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One of the first major documents, where the term ‘error’ has been used with this statistical connotation is, to the best of our knowledge, “Theoria Motus Corporum Celestium” by C. G. Gauss (1809). Since then, the term ‘error’ has commonly been used to signify a statistical estimate of the size of the difference between the measured and the true value of the measurand. Seminal books such as “Statistical Methods For Research Workers” by R. A. Fisher (1925) or “Inverse Methods For Atmospheric Sounding — Theory and Practice” by C. D. Rodgers (2000) publications by Gauss (1816), Pearson (1920), Fisher (1925), Rodgers (1990), and Mayo (1996) furnish evidence of this claim about the use of this term ‘error’. The esti-

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¹Possolo (2021) expresses this construal in more colorful words: “[...] Measurement uncertainty surrounds the true value of the measurand like a fog that obfuscates it, while measurement error is both the source of that fog and part and parcel of the measured value. Measurement uncertainty thus describes the doubt about the true value of the measurand, while measurement error quantifies the extent to which the measured value deviates from the true value.”

²When we use variances and standard deviations, we do not mean sample variances and sample standard variations but simply the second central moment of a distribution or its square root. In accordance with GUM-2008, this distribution can represent a probability in the sense of personal belief, and thus can include also systematic effects. See also Section 2.2.

³Other estimates are also used, e.g., robust ones like the interquartile range.

225 mated error is understood as a measure of the width of a distribution around the measured
~~(or estimated) value which tells the data user the probability — or the likelihood, depending~~
~~on the statistical framework used — density of a certain value to be measured or estimated~~
if the value actually measured ~~or estimated~~ was the true value. ~~Counterintuitively, in general~~
~~it does not always provide the probability density that the measured value is the true value.~~
~~This issue will be discussed in Section ??.~~ One might criticize equivocation of the traditional
230 language, but one can equally well consider this as a non-issue and trust that the context will
make clear what is meant. Often, some attributes are used for clarification and specification,
e.g., ‘probable error’ (Gauss, 1816; Bich, 2012), ‘statistical error’ Nuzzo (2014), ‘error estimation’
(Zhang et al., 2010) or ‘error analysis’ (Rodgers, 1990, 2000; Hughes and Hase, 2010).

~~In some cases, uncertainty scientists repudiate the use of the term ‘error’ to refer to a statistical~~
235 ~~estimate. Instead, they claim that the term ‘error’ only connotes the actual difference between~~
~~the measured or estimated and the true value of the measurand. E.g., on page 5 in GUM08,~~
~~the error of a measurement is the—~~ “More recently, GUM-2008 presented a narrower definition how
we have to conceive the term ‘error’ and have stipulated a new terminology, where the term
‘measurement uncertainty’ is used in situations where one would have said ‘measurement error’
240 before. According to GUM-2008, p.2, the uncertainty of a measurement is defined as “a parameter,
associated with the result of a measurement~~minus a true value of the measurand” (GUM08, p. 5).~~
~~It may be challenged that this definition is fully adequate, because it suppresses the use of the,~~
~~that characterizes the dispersion of the values that could reasonably be attributed to the measurand”.~~
Conversely, GUM-2008 (Annex B.2.19) allows for the term ‘error’ ~~for the statistical estimate of the~~
245 ~~actual error in the scientific literature since Gauss (1809) who uses the terms ‘error’ and only the~~
~~connotation ‘signed difference’, but their use of the terms ‘incertitudo’ (latin for error and uncertainty,~~
~~respectively) broadly as synonyms. The only difference, if any, in Gauss’ use of these terms seems~~
~~to be that error refers to estimated values, and uncertainty refers to the true values, saying that they~~
~~are not known with certainty.~~ error and error analysis’ in the first sentence of their 0.2 or ‘possible
250 error’ in their 2.2.4 only make sense if the statistical meaning of the term ‘error’ is conceded.

In spite of the explicit definition in ~~GUM08, there is~~ GUM-2008, there seems to be no unified
stance among ~~GUM08-GUM-2008~~ GUM-2008 endorsers as to what ‘error’ is. E.g., Merchant et al. (2017) uphold
that ‘error’ ~~connotes only the absolute~~ connotes only the signed difference, while Kacker et al.
(2007) or White (2016) refer to ‘error’ as a statistical estimate. Kacker et al. (2007) complain that
255 ~~GUM08-GUM-2008~~ GUM-2008 is often misunderstood, and we suspect that the cause for this might be that
~~GUM08-GUM-2008~~ GUM-2008 is indeed not sufficiently clear with respect to the differences between the ~~error~~
~~concept and the uncertainty concept~~ underlying error and uncertainty concepts.

~~Since the true value is not known, the actual difference between the measured or estimated~~
~~value and the true~~ The use of the term ‘uncertainty’ in GUM-2008 seems inconsistent: The general
260 GUM-2008 concept seems to be that the ‘error’ has to include all error sources and thus cannot

be known; ‘uncertainty’ is weaker, it is only an estimate of quantifiable errors, excluding the unknown components. This view is supported by the following quotation (GUM-2008, p. viii) “It is now widely recognized that, when all of the known or suspected components of error have been evaluated and the appropriate corrections have been applied, there still remains an uncertainty about the correctness of the stated result, that is, a doubt about how well the result of the measurement represents the value of the measurand cannot be calculated. This argument is often used to dispraise the traditional concept of error analysis as an obsolete concept. However, history and existing literature tell a different story. Above we have listed numerous sources where the term ‘error’ connotes a statistical quantity which can be estimated without knowledge of the true value of the measurand-quantity being measured.” It is not fully clear what this means. One possible reading is that they use the term ‘error’ in the redefined sense, viz., as a quantity which measures the actual deviation from the true value. Then this statement would be a mere truism, just saying that after all correction and calibration activities there is still a need for error (in the error concept terminology) estimation. The only other possible reading is that they want to say that, since due to unknown (unrecognized and/or recognized but not quantified⁴) error sources, error estimation will always be incomplete and there remains an additional uncertainty not covered by the error estimation. This often is very true but the use of the term ‘uncertainty’ would then be inconsistent in their document, because here the connotation of ‘uncertainty’ is the unknown (unquantified or even unrecognized) part of the error, which by definition cannot be assessed, while in the main part of their document, the connotation of ‘uncertainty’ seems to be a quantified statistical estimate. In summary, it is not clear if the ‘uncertainty’ includes the unknown error terms or not.

The introduction of the term ‘uncertainty of measurement’ seems to us a mere linguistic revision of an established terminology which does not connect to any further insights. The issue of whether the term ‘error’ should be used also for a statistical estimate cannot be judged on scientific grounds. It is a matter of stipulation, although in the main body of ~~GUM08~~ GUM-2008 this stipulation is presented as if it was a factual statement (“In this Guide, great care is taken to distinguish between the terms ‘error’ and ‘uncertainty’. They are not synonyms, but represent completely different concepts; they should not be confused with one another or misused.”, Sect 3.2, Note 2). The synonymy of ‘error’ and ‘uncertainty’ is thus neither true nor false but adequate or inadequate. Instead of quibbling about words we will, in the next Section, concentrate on the concepts behind these terms.

~~GUM08-~~

2.2 Conceptual Issues

Although GUM-2008 (Sect. 0.2) claims the “concept of uncertainty as a quantifiable attribute to be a relatively new concept in the history of measurement”, we uphold the view that it

⁴Rigorously speaking, within the concept of subjective probability recognized but unquantified uncertainties should not exist.

295 has long been recognized that the result of a measurement remains to some degree uncertain
even when a thorough measurement procedure and error evaluation is performed. Investigators
realized already in the 19th century that measurement results always have errors. Carl Friedrich
(Gauss, 1809) and Adrien-Marie Legendre (1805) formalized the required procedure of balancing
imperfect astrometric measurements by least squares fitting, in support of orbital calculations from
300 overdetermined data sets. And there is no reason to believe that earlier investigators were unaware of
the fact that they were not working on perfect observational data. Kepler's conclusion concerning the
elliptical shape of the orbit of Mars based on the rich observational dataset collected by Brahe would
have been impossible without proper implicit assumptions concerning the limited validity of the
reported values (Kepler, 1609). A rich methodological toolbox for error estimation and uncertainty
305 assessment has been developed since then, including systematic errors, error correlations, etc.

GUM-2008 does not only present traditional error analysis in a revised language but suggests
that there is more to it. That is to say, the entire concept is claimed to be replaced ~~-(see, e.g.,~~
GUM-2008, Sect 3.2.2., Note 2). We understand that ~~GUM08-GUM-2008~~ grants that the classical
concept of error analysis deals with statistical quantities, but these are statistical estimates of the
310 difference between the measured or estimated value and the true value. We take ~~GUM-GUM-2008~~
to be saying that the reference of even this statistical quantity to the true value poses certain problems,
because the true value is unknown and unknowable. As a solution of this problem, the uncertainty
concept is introduced which allegedly makes no reference to the true value of the measurand and is
thus hoped to avoid related problems. ~~GUM08-GUM-2008~~ (particularly Section 2.2.4) unfortunately
315 leaves room for multiple interpretations, but our reading is that an error distribution is understood
by GUM-2008 as a distribution whose ~~spread~~ dispersion is the estimated statistical error and whose
expectation value is the true value, while an uncertainty distribution is understood as a distribution
whose ~~spread~~ dispersion is the estimated uncertainty and whose expectation value is the measured
or estimated value.

320 ~~GUM08-GUM-2008~~ (p.5) characterizes error as “an idealized concept” and states that “errors
cannot be known exactly”. This is certainly true but it has never been claimed that errors can be
known exactly. Since not all relevant error sources are necessarily known, any error estimate remains
fallible but still it is and has always been the goal of error analysis to provide error estimates as
realistic as possible. To use the statistical conception of ‘error’ and conceding the fallibility of its
325 estimated value, it is not necessary to know the true value. It is only necessary to know the chief
mechanisms which can make the measured value deviate from the true value and to have estimates
available on the uncertainties of the input values to these mechanisms.

Some ~~GUM08-GUM-2008~~ endorsers (e.g., Kacker et al., 2007) try to draw a borderline between
error analysis and uncertainty assessment in a way that they associate error analysis with frequentist
330 statistics while uncertainty is placed in the context of Bayesian statistics. Frequentist statistics, we
understand, is a concept where the term ‘probability’ is defined via the limit of frequencies for

a sample size approaching infinity. This definition is ~~untenable~~ challenged because it involves a circularity: It is based on the large number theorem, according to which (strong version) a frequency distribution will almost certainly converge towards its limit. This limit is then associated with the probability. 'Almost certainly' means 'with probability 1'. The circularity is given by the fact that
335 the *definiendum* appears in the *definiens* (See, e.g., Stegmüller, 1973, pp. 27). Also the weak version of the large number theorem involves the concept of probability and thus poses a similar problem to the definition of the term 'probability'. We concede that many estimators in error estimation rely on frequency distributions. It is, however, a serious misconception to conclude from this that
340 error analysis is based on a frequentist definition of 'probability'. This is simply a *non sequitur*. Frequency-based estimators are consistent with any of the established definitions of probability, and their use does not allow any conclusion ~~on~~ about the definition of 'probability' in use.

3 ~~The definition of the term 'uncertainty'~~

~~According to GUM08, p.2, the uncertainty of a measurement is defined as "a parameter, associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand". GUM08 claims that this context does not make~~ The conceptual differences between error analysis and uncertainty analysis seem to come down to the different relations between the measured and the true value of the measurand. In GUM-2008 (p. 3 and p. 5), the claim is made that the uncertainty concept can be construed without reference to the 'true' value which is unknown anyway unknown and unknowable true value while the error concept cannot (GUM-2008, p.3), and that the uncertainty concept is more adequate because there can always exist unknown error sources which entail that an error budget can never be guaranteed to be complete (~~GUM08~~ GUM-2008 p. viii). It is stated that the uncertainty concept is not inconsistent with the error concept [~~GUM08~~ GUM-2008 p. 2/3]. There are, however, certain inconsistencies and shortcomings,
355 which are discussed in the following.

One of the major purposes of making scientific observations, besides triggering ideas on possible relations between quantities, is to test predictions based on theories on the real world (Popper, 1935). To decide if an observation corroborates or refutes a hypothesis, it is necessary to have an estimate how well the observation represents the true state, because it must be decided how well any
360 discrepancy between the prediction and the observation can be explained by the observational error (e.g., Mayo, 1996). Any concept of uncertainty ~~which~~ that is not related to the true state cannot serve this purpose.

On page 3, ~~GUM08~~ GUM-2008 says that the attribute 'true' is intentionally not used within the uncertainty concept because truth is not knowable. In ~~GUM08~~ GUM-2008, p. 59 it is claimed that
365 the uncertainty concept "uncouple the often confusing connection between uncertainty and the unknowable quantities "true" value and "error". The term 'measurand' in their definition, however,

is defined as the quantity intended to be measured (Joint Committee for Guides in Metrology (JCGM), 2009)~~(henceforth abbreviated GUM09); GUM08; GUM-2008~~, (p.32) says basically the same; ~~GUM09~~GUM-2009, p. 20, says that the ‘quantity’ is the same as the ‘true quantity value’.

370 Inserting this definition in the ~~GUM08~~GUM-2008 definition of uncertainty yields that, through the back door, uncertainty still refers to the true value. Thus it is not clear what the difference between the traditional concept of error analysis and the uncertainty concept is. Further, it is stated that systematic effects can contribute to the uncertainty. ~~GUM08~~GUM-2008 falls short of clarifying how a systematic effect be understood other than a systematic deviation between the measurement and
375 the true value ~~, the concept GUM08 that the concept GUM-2008~~ apparently tries to avoid. In order to justify the attribution of an uncertainty distribution to the systematic effects without relying on frequentist statistics, they invoke the concept of subjective probability. With this it becomes possible to assign an uncertainty distribution to the combined random and systematic uncertainty but still it is not clear how the systematic effect is defined without reference to the unknown truth.

380 Subjective probability reflects the personal degree of belief (~~GUM08~~GUM-2008, p. 39). Thus, a knowledge-dependent concept of probability is used in ~~GUM08. As discussed in the previous paragraph, this~~GUM-2008. This approach has been chosen to allow the treatment of systematic errors as dispersions, although the systematic error does not vary and cannot thus be characterized by a distribution in a frequentist sense (~~GUM08~~GUM-2008 p. 60). If we construe ‘estimated error’
385 and ‘estimated value’ as parameters of a distribution assigning to each possible value the probability (in a Bayesian context) or the likelihood (in a maximum likelihood context⁵) that it is the true value, no knowledge of the true value is required. This is because, by definition, the subjective probability distribution merely represents the knowledge of the person generating it. In ~~the context of subjective probability it is not clear why the true error or the true value should be needed to generate an error~~
390 ~~distribution. The values the rational agent believes~~GUM the error concept is discarded because the capability of conducting an error estimate allegedly depends on the knowledge of the true value. However, once having invoked the concept of subjective probability, no objective knowledge of the unknowable true value is needed any longer. The subjectivist can work with the value they beliefs to be true~~are sufficient in this case, because the error distribution does not tell us anything about the~~
395 ~~truth anyway but only about the agent’s believe of what truth is~~. This solves the alleged problem of the error concept, namely, that the true value is unknown.

There is nothing wrong with the subjectivist concept of probability, nor do we attack the possibility to combine random and systematic errors in a single distribution. This concept, however, makes the knowledge of the true value and the true error unnecessary, and still the estimated error can be
400 conceived as a statistical estimate of the absolute difference between the measured value and the true value. We consider it untenable and inconsistent to refer to the concept of subjective probability

⁵see Section ~~??~~3.2 for a deeper discussion of this issue.

when it comes in handy and to deny it when it would solve the conflict between the error and the uncertainty concepts.

Our skepticism about the possibility of dispensing with the concept of the true value is shared by, e.g., Ehrlich (2014), Grégis (2015), and Mari and Giordani (2014). Note that in the International Vocabulary of Metrology (known as VIM) (Joint Committee for Guides in Metrology (JCGM), 2012), although also issued by the JCGM, the concept and definition of the true value are explicitly retained.

~~The use of the term ‘uncertainty’ in GUM08 seems inconsistent. The general GUM08 concept seems to be that the ‘error’ has to include all error sources and thus cannot be known; ‘uncertainty’ is weaker, it is only an estimate of quantifiable errors, excluding the unknown components. This view is supported by the following quotation (GUM08, p. viii) “It is now widely recognized that, when all of the known or suspected components of error have been evaluated and the appropriate corrections have been applied, there still remains an uncertainty about the correctness of the stated result, that is, a doubt about how well the result of the measurement represents the value of the quantity being measured.” It is not fully clear what this means. One possible reading is that they use the term ‘error’ in the redefined sense, viz., as a quantity which measures the actual deviation from the true value. Then this statement would be a mere truism, just saying that after all correction and calibration activities there is still a need for error (in the error concept terminology) estimation. The only other possible reading is that they want to say that, since due to unknown (unrecognized and/or recognized but not quantified⁶) error sources, error estimation will always be incomplete and there remains an additional uncertainty not covered by the error estimation. This often is very true but the use of the term ‘uncertainty’ would then be inconsistent in their document, because here the connotation of ‘uncertainty’ is the unknown (unquantified or even unrecognized) part of the error, which by definition cannot be assessed, while in the main part of their document, the connotation of ‘uncertainty’ seems to be a quantified statistical estimate. In summary, it is not clear if the ‘uncertainty’ includes the unknown error terms or not.~~

In ~~GUM08, GUM-2008~~, p. 2/3 it is claimed that the concept of uncertainty “is not inconsistent with other concepts of uncertainty of measurement, such as a measure of the possible error in the estimated value of the measurand as provided by the result of a measurement [or] an estimate characterizing the range of values within which the true value of the measurand lies⁶ (VIM:1984 definition 3.09). Although these two traditional concepts are valid as ideals, they focus on *unknowable* quantities: the “error” of the result of a measurement and the “true value” of the measurand (in contrast to the estimated value), respectively. Nevertheless, whichever *concept* of uncertainty is adopted, an uncertainty component is always *evaluated* using the same data and related information...” (emphases in the original). It remains unclear how the concepts can, on the one hand, be consistent, while, on the other hand, it is claimed that the error approach and the uncertainty approach are actually conceptually

~~⁶Rigorously speaking, within the concept of subjective probability recognized but unquantified uncertainties should not exist.~~

⁶It is not clear how this can be achieved without explicit consideration of the Bayes theorem.

ally different and not only with respect to terminology. In GUM08, p.5 it reads “In this Guide, great care is taken to distinguish between the terms “error” and “uncertainty”. They are not synonyms, but represent completely different concepts; they should not be confused with one another or misused”

440 Since both concepts, however, are consistent, it is not clear in what the difference of the concepts consists.

Again, we come back to Kacker et al. (2007) who claim that error estimation and uncertainty analysis are best distinguished in the sense that the former relies on frequentist statistics while the latter is founded on Bayesian statistics. Here the following remarks are in order: (1) Many of the methods presented in GUM08, including their ‘Type A evaluation (of uncertainty)’, which is the ‘method of evaluation of uncertainty by the statistical analysis of series of observations’ are from the frequentist toolbox. (2) Kacker et al. (2007) endorse Monte Carlo methods to estimate uncertainty. Monte Carlo uncertainty estimation, however, is in its heart a frequentist method, because it estimates the uncertainty from the frequency distribution of the Monte Carlo samples. And (3) it is astonishing why GUM08, if representing a Bayesian concept, does not in the first place require to apply the Bayes theorem to convert the likelihood distributions into a posteriori probability distributions. The methodology proposed in GUM is uncertainty propagation. This is a mere forward (or direct) problem: given that x_{true} is the true value, and a measurement procedure with some error distribution, it returns a probability distribution for values x_{measured} that might be measured. However, GUM08’s definition of uncertainty “parameter, associated with *the result of a measurement*, that characterizes the dispersion of *the values that could reasonably be attributed to the measurand*” (emphasis added by us), seems associated with another meaning: given a measured value x_{measured} (“result of a measurement”) and a measurement procedure with some error distribution, what is the probability distribution x_{true} (“values that could reasonably be attributed to the measurand”). This is an inverse problem, for which Bayes theorem is applicable rather than uncertainty propagation.

460 Interestingly enough, early documents of the history of GUM (Kaarls, 1980; Bureau International des Poids et Mésures) provide evidence that the terminological turn from ‘error’ to ‘uncertainty’ was triggered only by linguistic arguments, based upon the fact that in common language the term ‘uncertainty’ is often associated with “doubt, vagueness, indeterminacy, ignorance, imperfect knowledge”. These early documents provide no evidence that ‘error’ and ‘uncertainty’ were conceived as two different technical terms [connotating](#) [connoting](#) different concepts. Any re-interpretation of the terms ‘error’ and ‘uncertainty’ as frequentist versus Bayesian terms or operational versus idealistic concepts came later.

470 In summary, it appears that the uncertainty concept is not essentially different from the error concept. We do, however, not claim that the terms ‘error’ and ‘uncertainty’ are fully equivalent; even in pre-GUM language there might be some subtle linguistic differences. We perfectly know our measurement (even if it is erroneous) and are ignorant with respect to and have imperfect knowledge only about the true value. This suggests that the uncertainty is an attribute of the true value while

475 ~~the error is associated with a measurement or an estimate. Because of the measurement error there is an uncertainty as to what the true value is. The uncertainty thus describes the degree of ignorance about the true value while the estimated error describes to which degree the measurement is thought to deviate from the true value. In this use of language both terms still relate to the same concept. This notion seems, as far as we can judge, to be consistent with the language widely used in the pre-GUM literature, but this issue deserves a more thorough linguistic assessment that is beyond the scope of~~
480 ~~this paper. The introduction of the term ‘uncertainty of measurement’ seems to us a mere linguistic revision of an established terminology which does not connect to any further insights.~~

~~In summary, we have to distinguish between two questions; first, whether the terms ‘error’ and ‘uncertainty’ have the same connotation, and second, whether the underlying concepts are indeed different. The first question is~~ The answer to the terminological differences was found to be contin-
485 gent upon the underlying stipulation, and that any statement about their equivalence or difference without reference to a definition is a futile pseudo-statement. The answer to the ~~second question~~ question of conceptual differences is less trivial and deserves some deeper scientific discussion. The main question still seems to be how the true value, the error or uncertainty, and the measured value are related with each other. This question will be addressed in the following sectionssection.

490 3 The unknown true value of the measurand

The alleged key problem of the error concept is, in our reading of ~~GUM08~~GUM-2008, that the value of the true value of the measurand is not known, and that this true value must appear neither in the definition of any term nor in the recipes to estimate it. To better understand this key problem, we decompose it into four sub-problems.

- 495 1. Quantities ~~of which whose~~ the value cannot be determined must not appear in definitions.
2. The error distribution must not be conceived as a probability density distribution of a value to be the true value.
3. Nonlinearity issues pose problems on error estimation if the true value is not known, at least in approximation.
- 500 4. ~~On~~One can never know that the uncertainty budget is complete because it can always happen that a certain source of uncertainty has been overlooked; thus, the full error estimate is an unachievable ideal and thus the estimated error does not provide a link between the measured value and the true value.

Some of these sub-problems are in some way formulated in ~~GUM08~~GUM-2008 but it is not exactly
505 specified there why the fact that the true value of the measurand is unknowable poses a problem to the ~~error scientist. Others have been formulated by us, serving, as arguments of the~~ scientist

applying traditional error estimation. We have formulated others as Devil's ~~advocate~~, ~~advocates~~,
which are intended to serve as working hypotheses ~~in order to moot to critically discuss~~ the error
and uncertainty concepts in the context of indirect measurements. In the following we will scrutinize
510 these theses one after the other.

3.1 The operational definition

~~GUM08~~~~GUM-2008~~ tries to avoid to use the true value of the measurand in the definition of the term
'uncertainty'. This strategy is employed because the true value of the measurand is "not knowable"
(~~GUM08~~~~GUM-2008~~, p. 3). It may be found puzzling why it should be necessary to know the value
515 of a quantity to use it in the definition of a term. The ~~weight of Thomas Bayes or the body height~~
~~of David Hume at a certain time~~ height of the Colossus of Rhodes or the Lighthouse of Alexandria
are well-defined quantities although we have no chance to measure them today⁷. Also we might
have a clear physical conception of what the temperature in the center of the sun might be although
it may not be practicable to put a thermometer there, and we even might not be able to figure out
520 any other, more sophisticated, method to assign an accurate observation-based value to this quantity.
Intuitively, we conceive the definition of a quantity and the assignment of the value to a quantity as
quite different things.

In ~~GUM08~~~~GUM-2008~~ it is claimed that the definition of 'uncertainty' is an operational one
(p. 2). An operational definition defines a quantity by stipulating a procedure by which a value
525 is assigned to this quantity. The concept of operational definitions was suggested by Bridgeman
(1927) in order to give terms in science a clear-cut meaning. This operationalism, at least a narrow
conception of it, has its own problems, has received considerable criticism and has led to deep
philosophical discussions (see, e.g., Chang, 2019). To summarize these is beyond the scope of this
paper and for here it must suffice to mention that there are alternatives, such as theoretical definitions
530 or reduction of the *definiendum* to previously defined terms.

~~GUM08~~~~GUM-2008~~'s claim that the uncertainty concept is based on an operational definition leads
to two further inconsistencies. First, no unambiguous operation is stipulated on which the definition
can be based, but multiple operations are proposed, which might give different uncertainty estimates.
Thus, the definition is void. Our critical attitude with respect to operationalism in the context of ~~GUM~~
535 ~~GUM-2008~~ is shared, e.g., by ~~MARH4~~~~Mari and Giordani (2014)~~.

The other problem with the operational definition is the following: In ~~GUM08~~~~GUM-2008~~, pp
2-3, it is claimed that the uncertainty concept is not inconsistent with the error concept, and a few
lines later it reads "an uncertainty component is always *evaluated* using the same data and related
information."⁷ (emphasis in the original). The latter suggests that within the error concept the same
540 operations are used as within the uncertainty concept. Since the operations define the term and the
related concept, the uncertainty concept and the error concept must be the same.

⁷We owe this illustrative example to Possolo (2021).

In summary, the fact that the true value of the measurand is unknowable is a problem for the definition of the term ‘error’ and its statistical estimates only if we commit ourselves to the doctrine of that only operational definitions must be used. If we abandon this dogma, there is nothing wrong with conceiving the estimated error as a statistical estimate between the measured or estimated and the true value, and the problem is restricted to the assignment of a value to this quantity. Related issues are investigated in the following.

3.2 ~~Likelihood, probability, and the base rate fallacy~~ Measurements as inverse processes

~~The argument discussed in the following is not put forward in GUM08, probably because there indirect measurements are not in the focus. However, since we do focus on indirect measurements, and since this argument is of particular importance in this context, we present and discuss this argument.~~ Many conceptions of measurement models exist, which relate the measured value to the true value, and depending on the context, one can be more adequate than another (Possolo, 2015). GUM-2008 recommends a model that conceives the estimate of the true value of the measurand as a function of the measured value. Since in remote sensing of the atmosphere multiple atmospheric states can cause the same set of measurements, and the measurement function thus would be ambiguous, we prefer a different concept, as outlined in the following.

The causal error points from the true value to the measured signal. Thus, the estimation of the true value from a measured value can be conceived as an inverse process. An argument along this line of thought, but in a context wider than that of remote sensing of the atmosphere, has been put forward by Possolo and Toman (2007). The inverse characteristic of the estimation problem is particularly true for indirect measurements, e.g., remote sensing, but direct measurements can easily be conceived as indirect measurements. When reading the thermometer, we actually read the length of the mercury column (the measured value), apply inversely the law of thermal expansion, and get an estimate of the temperature. In trivial cases, when a measurement device has a calibrated scale from which the target quantity can be directly read, the inverse process is effectively pretabulated in the scale. Only in these cases the measured value and the estimate of the measurand are the same.

~~Counterintuitively, in general, the error bar or the uncertainty estimate do not tell the probability density of the true value with respect to the measured value, unless the prior probability~~ With a transfer function F available, that approximately describes the process that links the true value x of interest to the measured value, the expected measured signal $y_{\text{expected}} = F(x)$ can be estimated. The distribution of the measurement error around y_{expected} describes the probability of ~~the measurand has been considered in the retrieval~~ any value y to be measured.

Conversely, for a given measurement y_{measured} , the inversion of the transfer function allows to estimate the true value x . If a genuine inversion of the transfer function is not possible due to ill-posedness of the inverse problem in the sense of Hadamard (1902), workarounds like least squares methods or regularized inversion schemes are available (see, e.g., von Clarmann et al. 2020 for a

summary of some methods of particular relevance for remote sensing). Counterintuitively, however, in general, neither the estimate will be the most probable value of x , nor will the mapping of the measurement error distribution into the x -space yield the probability distribution of any value to be the true value. This holds even if the error distribution is extended to include also systematic effects, and if all error correlations are adequately taken into account in the case of multi-dimensional measurements. It is the theorem of Bayes (1763) which makes the difference. The only inverse scheme where such a probabilistic interpretation is valid in the x -space, is the maximum a posteriori method (Rodgers, 2000), which employs a Bayesian estimator.

The non-consideration of the Bayes theorem goes under the name of ‘base rate fallacy’. 50% of people suffering Covid-19 have fever (Robert Koch Institut, 2020), but this does not imply that the probability is 50% that a person suffering fever to have with fever has Covid-19 is 50%. To estimate the latter probability requires knowledge of the percentage of people being infected with the Corona virus, and the probability that a person suffers fever for any reason. In metrology the situation is quite analogous. If we define the true value to be x , the ideally measured value $f(x) = y_{\text{ideal}}$, and the estimated measurement error in terms of the standard deviation σ_y , then the probability density of a certain value y to be measured is given by a pdf with y_{ideal} mean and σ_y spread. This, however, does not imply that, if we measure y with an uncertainty of σ_y , and propagate σ_y through the inversion procedure to get the uncertainty of \hat{x} , namely, σ_x , that the probability of some x being the true value of the measurand is given by the pdf with mean \hat{x} and σ_x spread. Again, it is the a priori probability distribution⁸ which is missing. There are three ways out of possible solutions to cope with this problem. For now we will defer the problem of a possibly incomplete error budget to Section 3.4 and assume that the error budget is complete.

The first solution is to apply a Bayesian retrieval scheme. Indeed in many cases, the solution of the inverse problem \hat{F}^{-1} employs retrieval scheme that is based on a Bayesian estimator. Examples are found, e.g., in Rodgers (2000) or von Clarmann et al. (2020). On the supposition that the error budget is complete, the interpretation of the error bar as the spread dispersion of a distribution representing the probability density that a certain value is the true value is correct.

The second solution is the application of the principle of indifference. Often The problem with this approach is that often there is no firm a priori knowledge on the value of the measurand available. Gauss (1809) solves this problem by

The second solution is the application of the indifference principle. principle of indifference, as applied, e.g., by Gauss (1809). That is, the same a priori probability is assigned to all possible values of the measurand. With this, e.g., in the application to a linear inverse problem and normal distributions of uncertainties, the Bayesian solution collapses back to a simple unconstrained least squares solution. Due to the assumption of the equidistribution of the a priori probabilities, the estimated uncertainty of the estimate can still be interpreted as the width of the probability density

⁸or the background frequency distribution.

function of the true value of the measurand. This concept of ‘non-informative a priori’, however,
615 has its own problems. Even if we ignore some more trivial problems for the moment, e.g., that
some quantities cannot, by definition, take negative values, this concept can lead to absurdities:
If we assume that we have no knowledge on, say, the volume density of small-particle aerosols
in the atmosphere, and describe this missing knowledge by an equidistribution of probabilities, this
620 would correspond to a non-equidistribution of the surface densities, due to the non-linear relationship
between surface and volume. It strikes us as absurd that information can be generated just by such a
simple transformation from one domain into another. The principle of indifference, upon which the
concept of non-informative priors is built, is critically but still favorably discussed, e.g., by Keynes
(1921, Chapter IV). The concept of non-informative priors is still criticized even in the Bayesian
community (e.g. D’Agostini, 2003).

625 The third solution is the likelihood interpretation, which has been introduced by Fisher (1922).
The likelihood that the true value is x if the measured signal is y equals the probability density
that y is measured if the true value is x . No prior information is considered. Solution of the inverse
problem ~~\hat{F}^{-1}~~ by maximizing the likelihood of x does not provide the most probable estimate of
 x , and accordingly the error bar of the solution must not be interpreted as the width of a probabil-
630 ity distribution of the true value. Application to a linear inverse problem and normal distributions of
uncertainties renders formally the same estimator as the Gaussian least squares solution, but its inter-
pretation has changed. It can no longer be interpreted as the ~~mean of a pdf~~ maximum of a probability
density function of the true value. ~~The distribution with mean \hat{x} and the standard deviation σ_x can~~
~~still be interpreted as a likelihood distribution of the true value around the estimate.~~ If need be, in
635 some cases, i.e., if the inverse problem is well-posed enough to allow an unconstrained solution, the
maximum likelihood estimate can be, *post factum*, transformed into a Bayesian estimate by applica-
tion of the Bayes theorem. ~~Certainly one might urge the objection that the true a priori distribution~~
~~is never known and that this transformation is thus an idealized concept~~

We concede that the interpretation of a measured value as the most probable true value is
640 problematic. This implies that also the interpretation of the error estimate as the width of a
distribution around the true value is not generally valid. These problems could justify some
reluctance with regard to the concept of the true value. This argument ~~is only applicable by~~
~~the frequentist statistician. The uncertainty statisticians, involving the base-rate fallacy,~~ how-
ever, ~~have already committed themselves to the concept of subjective probability because~~
645 ~~otherwise the aggregation of random uncertainties and systematic effects into one combined~~
~~uncertainty distribution would not be possible⁸. For the uncertainty statistician the a priori~~
~~distribution represents the knowledge of a rational agent, as opposed to the true underlying frequency~~
~~distribution. Thus this counter-argument is not valid.~~ is not invoked in GUM-2008.

⁸~~Willink and White (2012) might contradict here and make the claim that even systematic errors can be conceived as drawn from some population of errors, consistent with the frequentist view.~~

~~In summary~~ Some interpretations of GUM-2008 (e.g. White, 2016; Kacker et al., 2007) associate
650 it with a Bayesian conception of probability and seem to suggest that error estimation and uncertainty
analysis are best distinguished in the sense that the former relies on frequentist statistics while the
latter is founded on Bayesian statistics. Thus one might suspect that ‘uncertainty’ is simply the
Bayesian replacement of error. Here the following remarks are in order:

(1) Many of the methods presented in GUM-2008, including their ‘Type A evaluation (of
655 uncertainty)’, which is the ‘method of evaluation of uncertainty by the statistical analysis of series
of observations’ are from the frequentist toolbox. Gleser (1998) find that the methods suggested in
GUM are neither fully frequentist nor fully Bayesian. Further, it is ~~true that the error bar does not
necessarily represent the width of a distribution representing the probability density that a certain
value is the true value of the measurement. In the case of a Bayesian estimate~~ not quite clear which
660 of Bayes’ methods and principles a scientist has to use to be a Bayesian (c.f., e.g., Fienberg, 2006),
since the Bayes theorem is accepted also by non-Bayesians, and the use of maximum likelihood
methods, introduced by the almost ‘militant’ frequentist R. A. Fisher (1922) does, as far as we can
judge, not commit one to use a frequentist definition of the term ‘probability’. The GUM-2008 does
not provide a clear reference to a specifically Bayesian uncertainty analysis method. GUM-2008
665 makes reference to Jeffreys (1983) as an authority of the degree-of-belief-concept of probability.
~~Jeffreys, however, which is quite frequently chosen in remote sensing, it can be conceived this way.
And in~~ offers no clue as to what the difference between ‘error’ and ‘uncertainty’ might be. In the
context of ~~maximum likelihood estimates, measurements or observations, Jeffreys always uses the
term ‘error’ (e.g., op. cit., p. 72), and often we find statements like “the probable error [...] is the
670 uncertainty usually quoted” (op. cit., p. 72), “no uncertainty beyond the sampling errors” (op. cit.,
p. 389), or “treat the errors as independent” (op. cit., p. 443). With the statement that errors are
not mistakes (op. cit., p. 13), Jeffreys explicitly contradicts the GUM pioneers (Kaarls, 1980) and
GUM-2008 endorsers Merchant et al. (2017). Also Press (1989) is referenced by GUM-2008 only
to defend the use of a subjective concept of probability but not in a context aiming at the clarification
675 of the alleged difference between ‘error’ and ‘uncertainty’.~~

(2) If the uncertainty concept was indeed founded on a Bayesian framework, it would be
astonishing why it does not in the first place require to apply the Bayes theorem to convert the
likelihood distributions into a posteriori probability distributions. The methodology proposed in
GUM-2008 is uncertainty propagation. This is a mere forward (or direct) problem: given that x_{true} is
680 the ~~estimated error still can be conceived as the width of a distribution representing the likelihood that
a certain value is the true value of the measurement,~~ and a measurement procedure with some error
distribution, it returns a probability distribution for values x_{measured} that might be measured. However,
GUM-2008’s definition of uncertainty “parameter, associated with *the result of a measurement, that
characterizes the dispersion of the values that could reasonably be attributed to the measurand*”
685 (emphasis added by us), seems associated with another meaning: given a measured value (“result of

a measurement”) and a measurement procedure with some error distribution, what is the probability density distribution of the “values that could reasonably be attributed to the measurand” to be the true one. This is an inverse problem, for which Bayes theorem is applicable rather than uncertainty propagation.

690 (3) Interestingly enough, Willink and White (2012), who use the term ‘uncertainty’ also in a frequentist framework, report that the turn to the new terminology happened already in 1980/81, and make a strong case that various allegedly purely Bayesian concepts of GUM-2008 can be given a valid frequentist interpretation. ~~All these statements, however, are contingent upon the assumption that the error or uncertainty budget is complete. This problem is deferred to Section 3.4.~~

695 Thus, we reject the hypothesis that uncertainty concept as presented in GUM-2008 is a Bayesian concept. Bayesianism does not help to understand the claimed differences between the error concept and the uncertainty concept.

3.3 Nonlinearity issues

The uncertainty concept relies on the possibility of evaluating uncertainties caused by measurement errors and “systematic effects” without knowledge of the true value. This is certainly granted
 700 for linear problems. ~~The resulting uncertainty in \hat{x} , namely $\sigma_{\hat{x}}$, generated by a measurement error statistically characterized by its standard deviation σ_y or by a systematic effect, e.g., a not exactly known value of a constant b will be the same for each \hat{x} , and no specific relation between the estimate \hat{x} and the true~~ Here the uncertainty estimates do not depend on the value of the measurand x is
 705 ~~required.~~ This is because in the linear case Gaussian error propagation holds,

$$\sigma_{x,\text{noise}}^2 = \left(\frac{\partial x}{\partial y} \right)^2 \sigma_{y,\text{noise}}^2,$$

and

$$\sigma_{x,b}^2 = \left(\frac{\partial x}{\partial b} \right)^2 \sigma_b^2,$$

or for vectorial quantities

710 $\mathbf{S}_{x,\text{noise}} = \mathbf{G} \mathbf{S}_y \mathbf{G}^T,$

and

$$\mathbf{S}_{x,b} = \mathbf{G}_b \mathbf{S}_b \mathbf{G}_b^T,$$

where $\mathbf{S}_{x,\text{noise}}$, $\mathbf{S}_{x,b}$ and \mathbf{S}_b are the covariance matrices and \mathbf{G} and \mathbf{G}_b the matrices of partial derivatives $\frac{\partial x_n}{\partial y_m}$ and $\frac{\partial x_n}{\partial b_k}$, respectively⁸.

⁸For standard deviations and covariance matrices we use the notation suggested by von Clarmann et al. (2020) where the first subscript indicates the domain to which the uncertainty or error estimate refers, and the optional second subscript the source of the error.

715 For nonlinear problems the situation is more complicated because ~~Equations (??) to (??) are~~
~~only valid~~ Gaussian error propagation is valid only in approximation. ~~The error scientist can invoke~~
Within the concept of error propagation, the concept of moderate nonlinearity (Rodgers, 2000) can
be invoked. That is to say, \hat{x} the estimated value of the measurand is assumed to be a reasonably good
720 approximation of ~~the measurand~~, and the partial derivatives needed for Gaussian error estimation
are evaluated at this estimate. If the ~~error in b is small enough ensuring that the resulting $\hat{x} \pm \sigma_x$ is~~
~~within~~ resulting error bars are small enough to ensure that the range covered by the interval defined
by the estimated value plus minus the error bar is confined to the range where linear approximation
is justifiable, then ~~σ_x is a~~ the error estimates are, while less-than-perfect ~~but far better than useless~~
~~estimate of the corresponding error component in x , still far better than useless~~.

725 The ~~uncertainty scientist~~ endorser of the uncertainty concept has a problem if they want to stay
consistent with their doctrine. Since knowledge of ~~the true value~~ is denied, ~~the approximation~~
 ~~$\hat{x} \approx x$ begs justification and~~ it is not clear how Gaussian error estimation can be applied to ~~systematic~~
~~effects~~ the propagation of uncertainties, because it is not clear for which value of the measurand the
required partial derivatives shall be evaluated.

730 On the face of it, Monte Carlo error estimation or other variants of ensemble-based sensitivity
studies can serve as an alternative. These, however, also invoke the nonlinear model ~~F and results~~
that links the measured signal with the measurand, and uncertainty estimates thus still depend on
the choice of ~~\hat{x} , and the estimate that represents the true value;~~ any choice of this value which is
not closely related to the true value ~~of the measurand~~ will produce uncertainty estimates which
735 are recalcitrant against any interpretation. Monte Carlo and related methods, however, are apt for
the ~~error scientists to estimate~~ estimation of the error budget including the systematic effects if f is
too nonlinear to justify Gaussian error estimation, if the approximate knowledge of the measurand
is conceded.

In summary, the evaluation of uncertainties in the case nonlinearity poses a problem to the
740 ~~uncertainty scientist~~ scientist who denies the approximate knowledge of the true value of the
measurand, because the uncertainty estimate depends on the assumed value of the measurand, and
 ~~$\hat{x} \approx x$ it~~ must be assumed that it represents the true value reasonably well. Within the framework of
error analysis this assumption is allowed and measurement errors as well as systematic effects thus
can be evaluated also for nonlinear inverse problems.

745 3.4 Incompleteness of the error budget

The arguments put forward above are based on the supposition that the error budget is complete.
Beyond measurement noise, the total error budget includes systematic effects in the measured sig-
nal~~y~~, uncertainties in parameters both than the measurand that affect the measured signal, and
effects due to the chosen inverse scheme \tilde{F}^{-1} . If our reading of ~~GUM08~~ GUM-2008 is correct, then
750 the most severe criticism by GUM-2008 of the ‘error concept’ ~~by GUM08~~ is that one can never be

sure that the error budget is indeed complete, and that thus the ~~distribution with \hat{x} expectation and σ_x standard deviation cannot tell us the probability density that any value of x is the error estimate~~ does not characterize the difference between the value estimated from the measurement and the true value.

755 The precision of a measurement is a well-behaved quantity in a sense that it is testable in a straight forward way: ~~From~~ from at least three sets of collocated measurements of the same quantity, where each set is homogeneous with respect to the expected precision of its measurements, the variances of the differences provide ~~unambiguous precision estimates~~ unambiguous precision estimates (see, e.g., McColl et al. 2014 or Stoffelen 1998). The situation is more difficult for biases. Biases between
760 different measurement systems do not tell us what the bias of one measurement system with respect to the – unfortunately unknowable – truth is. Even if the number of measurement systems is quite large, it is not guaranteed that the mean bias of all of them is zero. And an infinite number of measurement systems is out of reach in a real world. Up to that point we concede that a positive proof of the completeness of the error budget is impossible. But this is not the end of the story.

765 A falsificationist (Popper, 1935) approach is more promising. It follows the rationale that it will never be possible to prove that our assumptions on the bias of a measurement system is correct. Instead, we estimate the bias as well as we can, and use it as a best estimate of the bias until some test provides evidence that the estimate is incorrect. Such a test typically consists of the intercomparison of data sets from different measurement systems. If the bias between these data sets is larger than
770 the combined systematic error estimates, at least one of the systematic error estimates is too low and has to be refuted. Further work is then needed to find out which of the measurement systems is most likely to underestimate its systematic error. Conversely, as long as the mean difference of the measurements of the same measurand can be explained by the combined estimate of the systematic errors of both measurement systems, the systematic error estimates can be maintained, although
775 this is, admittedly, no proof of the correctness of the error estimates. But as long as severe tests as described above are executed and the error estimates cannot be refuted, it is rational to believe that they are sufficiently complete.

~~We have mentioned above that the uncertainty concept depends on the acceptance of the subjective probability in the sense of degree of rational belief. Without that, an error budget including systematic effects would make no sense because systematic effects cannot easily be conceived as probabilistic in a frequentist sense; that is to say, the resulting error cannot be conceived as a random variable in a frequentist sense. Being forced to adopt the concept of probability as a degree of rational belief, it makes perfectly sense to conceive, after consideration of the Bayes theorem (see Section ??) the distribution with expectation \hat{x} and covariance $\sigma_{x,\text{total}}$ as the probability distribution which
785 tells the rational agent the probability of any value to be the true value.~~

4 The applicability of ~~GUM08~~ GUM-2008 to remote sensing of the atmosphere

In this Section we identify issues where ~~GUM08~~ GUM-2008 clashes with the needs of error or uncertainty estimation in the field of remote sensing of atmospheric constituents and temperature. These issues are (1) since the atmospheric state varies quasi-continuously in space and time, the measurand is not well defined; and (2) there are applications of atmospheric data where the total uncertainty estimate alone does not help; ~~(3) Eq 11 in GUM08 is in conflict with the causal arrow, and (4) some GUM08 interpretations commit one to Bayesianism but some assumptions Bayesianism is based on cannot be logically inferred from generally accepted axioms nor traced back to observations.~~

4.1 What if the measurand is not well-defined?

On macroscopic scales, atmospheric state variables vary continuously in space on time. On microscopic scales, the typical target quantities, concentrations or temperature, are not even defined. A typical example of this problem is ~~be~~ the volume mixing ratio (VMR) of a certain species at a point in the atmosphere (See also, von Clarmann, 2014). The determination of a quantity like this requires a canonical ensemble of air but in the real, ~~inhomogenous~~ inhomogeneous, atmosphere, this quantity does not exist. It is an uninstantiated ideal. Due to these inhomogeneities the air volume sounded must be infinitesimally small, i.e., it must approach a point. In the real atmosphere there is either a target molecule at this point (VMR = 1) or another molecule (VMR = 0) or no molecule at all (undefined VMR due to division by zero). Thus, one measures only averages over finite inhomogeneous air volumes. This approach, supposedly the only possible approach, clashes with the premise of ~~GUM08~~ GUM-2008⁸ that the measurand needs to be well defined. Measuring atmospheric state variables requires the specification of the region the average is made over. The relevant toolbox of atmospheric data characterization includes concepts like resolution, averaging kernels etc. (see Rodgers, 2000 for detail). Since this type of measurements is apparently out of the scope of ~~GUM08~~ GUM-2008, the latter is ~~very quite~~ silent with respect to solutions to the problem of the characterization of measurements of quantities that are not well defined. Broadening the scope and applicability of the ~~GUM08~~ GUM-2008 framework to include less than ideally defined measurands and measurements that demand inverse methods would significantly increase the value and utility of ~~GUM08~~ GUM-2008 approach. Relevant recommendations on data characterization developed within the TUNER activity (von Clarmann et al., 2020) aim at helping to reach this goal.

4.2 The combined error

One of the positive aspects of ~~GUM08~~ GUM-2008 is that it breaks with the misled concept of characterizing systematic errors with 'safe bounds' (Kaarls, 1980; Kacker et al., 2007; Bich, 2012). This

⁸In ~~GUM08~~ GUM-2008 this problem is recognized but no solution is offered; ~~since~~; the term 'definitional uncertainty' is introduced in this context but not applied in practice.

concept was sometimes endorsed by error statisticians subscribing to frequentism. Within a frequentist concept of probability, a probabilistic treatment of systematic errors was not easily possible because due to its systematic nature a systematic error cannot easily⁹ be characterized by a frequency or probability distribution. The concept of subjective probability solves this problem. With the subjectivist's toolbox, it is no longer a problem to assign probability density functions, standard deviations and so forth when characterizing systematic errors. This possibility is a precondition for aggregating systematic and random errors to give the total error. ~~GUM08~~GUM-2008, however, goes a step further and even denies the necessity to report random and systematic errors independently. Here we have to urge severe objections.

Von Clarman et al. (2020) explicitly ~~demand~~recommend that error estimates be classified as random or systematic ;¹⁰ . In contrast, ~~GUM08~~GUM-2008 (E.3.3 / E3.7) state: "In fact, as far as the calculation of the combined standard uncertainties [...] is concerned, there is no need to classify uncertainty components and thus no real need for any classificational scheme." If indeed meant as written, we challenge the claim that a total combined error budget is sufficient and therefore no classificational scheme is needed at all. Characterizing the measurement of a unique quantity, e.g. the value of a natural constant ~~ageed~~agreed upon by the calibration authorities, by a single error margin might be sufficient. But most measurements, and particularly those of atmospheric state variables such as temperature, concentrations of trace species, and so forth, deal with quantities varying as function of time and space. Any sensible use of the resulting datasets requires a clear distinction between statistical and systematic error budgets. For example, for time series analysis targeted at the determination of trends, the total error budget is of no use but the random error budget is needed instead. This is because any purely additive systematic error component cancels out in this application and ~~their consideration would unduly~~its consideration in the error budget would unduly distort the weights of the data points available. In summary, the denial of the importance of distinguishing between random errors and systematic errors does not provide proper guidance, and altogether is a strong ~~misjudgement~~misjudgment. The data users must be provided with all information required to tailor the relevant error budget to the given application of the data.

Benevolent readers of ~~GUM08~~GUM-2008 take the GUM authors to be saying only that the aggregation of estimated errors to give the total error budget follows the same rules for systematic and random errors, and that the criticized statement is not meant to deny the importance of distinguishing between random and systematic errors beyond the mere aggregation process. If this reading is correct, we agree, but here ~~GUM08~~GUM-2008 leaves room for interpretation.

⁹The qualification not *easily* was chosen because frequentists still might sample over multiple universes or apply other measures to squeeze systematic errors in a frequentist concept.

¹⁰In this context it is important to note that, in contrast to some older conceptions, von Clarman et al. (2020) define 'systematic errors' as bias-generating errors and 'random errors' as variance-generating errors. To avoid confusion with the older conceptions, one can use instead the descriptive terms 'persistent' and 'volatile' errors as suggested by Possolo (2021). This is not done here to maintain consistency with von Clarman et al. (2020).

850 4.3 The causal arrow

Putting quantum effects aside, the measured value is unambiguously determined by the true value via causal processes plus a given but unknown error term. In other words, the causal arrow points from the atmospheric state to the measured value. The inverse direction, however, is ambiguous even if we set the problem of the unknown error term aside. Many atmospheric states can cause the same measurement, even if the noise term is exactly the same. Thus, it seems more adequate to us to formulate the measurement problem using a function that describes the measurement as a function of the atmospheric state, viz., the measurand (Eq. ??) and not vice versa. The inverse measurement equation (Eq. ??) which describes the dependence of the atmospheric state variables on the measured values is not unambiguously defined because the regularization term used to solve ill-posed problems or the norm chosen to provide a solution for over-determined problems, all contained in the term \tilde{F}^{-1} , are based on assumptions, depend on the personal preference of the person performing the inversion, and need to be considered in the data and uncertainty characterization (See, Section ??, and von Clarmann et al., 2020 for details). Thus we think that it is essential to appreciate the inverse nature of the problem, and this is much easier if the measurement equation describes the forward problem and thus does not suggest an unambiguous determination of the measurand from the measured quantity. An argument along this line of thought, but in a context wider than that of remote sensing of the atmosphere, has been put forward by Possolo and Toman (2007).

4.3 Bayesian versus non-Bayesian

Some interpretations of GUM08 (e.g. White, 2016; Kacker et al., 2007) associate it with a Bayesian conception of probability. Thus one might suspect that ‘uncertainty’ is simply the Bayesian replacement of error. But things seem not to be so simple, for two reasons.

(1) It is, however, not quite clear which of Bayes’ methods and principles a statistician has to use to be a Bayesian (c.f., e.g., Fienberg, 2006), since the Bayes theorem is accepted also by non-Bayesians, and the use of maximum likelihood methods, introduced by the almost ‘militant’ frequentist R. A. (Fisher, 1922) does, as far as we can judge, not commit one to use a frequentist definition of the term ‘probability’.

(2) Interestingly enough, Willink and White (2012) use the term ‘uncertainty’ also

5 Conclusions

We have mentioned above that the uncertainty concept depends on the acceptance of the subjective probability in the sense of degree of rational belief. Without that, an error budget including systematic effects would make no sense because systematic effects cannot easily be conceived as probabilistic in a frequentist framework, report that the turn to the new terminology happened already

in 1980/81, and make a strong case that various allegedly purely Bayesian concepts of GUM08 can be given a valid frequentist interpretation.

885 Defenders of the doctrine that ‘error’ and ‘uncertainty’ have a different meaning and that the error concept is a purely frequentist concept may refer to (Mayo, 1996, Ch. 13), who admits to frequentism and indeed proposes an ‘error-statistical philosophy of science’. Her concept, however, does not deal with measurement errors but with errors in the acceptance or rejection of hypotheses. Thus it cannot be interpreted in a way that measurement errors are a purely frequentist concept.

890 In the community of remote sensing, both maximum likelihood and Bayesian retrieval schemes are in use. Depending on the measurement type and the anticipated use of the data both have their pro’s and con’s. In order to avoid to make the rift between the Bayesian and non-Bayesian¹¹ part of the community even worse, the TUNER consortium has decided not to make a recommendation as to which of these retrieval schemes is thought to be superior. It was considered as more important to provide an adequate scheme for error or uncertainty estimation for any of these retrieval approaches. 895 As a consequence, it is not considered as adequate to custom-taylor uncertainty reporting to the Bayesian philosophy.

White (2016) reports that paradoxes shatter the bedrocks of Bayesian philosophy, namely the likelihood principle that says that all relevant evidence about an unknown quantity obtained from 900 an experiment is contained in the likelihood. Others accept the theoretical validity sense; that is to say, the resulting error cannot be conceived as a random variable in a frequentist sense. Being forced to adopt the concept of probability as a degree of rational belief, it makes perfectly sense to conceive, after consideration of the Bayes theorem but challenge its applicability in real life because of the unknown and unknowable prior probabilities. It has been recognized by 905 (Hume, 2003/1739, 1748) that what was valid yesterday might not be valid tomorrow. This implies that a statistic of past events might not provide a reliable prior for future inverse problems. Also the use of the so-called non-informative prior can be challenged: The domain in which the prior is expressed is an ad-hoc decision and any non-linear transformation will render an informative prior. E.g., a flat, thus apparently non-informative velocity distribution goes along with a non-flat, thus 910 informative distribution of kinetic energy. Similarly an equidistribution of droplet diameters goes along with a non-flat, thus informative, distribution of droplet volumes, etc. This is considered by some as an absurdity brought about by the concept of non-informative priors.

More generally speaking, the Bayesian philosophy relies on a couple of unwarranted assumptions, e.g., the likelihood principle and the indifference principle. The proof of the former has been 915 challenged (Evans, 2013; Mayo, 2013, quoted after White, 2007), and the latter has been criticized

¹¹We challenge the dichotomy ‘Bayesian vs. frequentist’. Not every non-frequentist is a whole-hearted Bayesian; not all objective probabilities are frequentist (see, e.g. Popper, 1959). Not everybody who endorses a subjective concept of probability accepts Bayesian tenets on confirmation theory and test theory. Further, subjectivist and objectivist probability concepts are not necessarily in contradiction but can be bridged (Lewis, 1980).

as not deducible from any accepted axioms. Thus a pro or contra Bayesian decision is a purely philosophical decision, and it does not seem adequate to make such a decision generally binding.

While it is fully agreeable that the concept of error reporting has long relied and still relies on a subjective, i.e., information-dependent concept of probability¹¹, this does not commit one to accept

920 Bayesianism in full.

Coming back to the title question whether error and uncertainty are different things and different concepts, and accepting that traditional error analysis was compatible with a degree-of-belief conception of probability, we are left with two possible interpretations. One is that it is only the endorsement of a subjective concept of probability that allegedly makes uncertainty analysis

925 Bayesian and defines the difference between error and uncertainty. If so, we raise the objection that classical error analysis is not a purely frequentist approach. The other interpretation is that there is more to it, and Bayesian uncertainty analysis is indeed something entirely different. The GUM08 does not provide a clear reference to such a Bayesian uncertainty analysis method. GUM08 makes reference to Jeffreys (1983) as an authority of (see Section 3.2) the distribution with expectation

930 \hat{x} and covariance $\sigma_{x,\text{total}}$ as the degree-of-belief concept of probability. Jeffreys, however, offers no clue as to what the difference between ‘error’ and ‘uncertainty’ might be. In the context of measurements or observations, Jeffreys always uses the term ‘error’ (e. g., op. cit., p. 72), and often we find statements like “the probable error ... is the uncertainty usually quoted” (op. cit., p. 72); “no uncertainty beyond the sampling errors” (op. cit., p. 389), or “treat the errors as independent” (op. cit., p. 443). With the statement that errors are not mistakes (op. cit., p. 13), Jeffreys explicitly

935 contradicts the GUM pioneers (Kaaris, 1980) and GUM08 endorsers Merchant et al. (2017). Also Press (1989) is referenced by GUM08 only to defend the use of a subjective concept of probability but not in a context aiming at the clarification of the alleged difference between ‘error’ and ‘uncertainty’. probability distribution which tells the rational agent the probability of any value to be
940 the true value.

We concede that Bayesians and frequentists may use the error or uncertainty estimates in a different way. In situations where a hypothesis is to be tested on the basis of measurement data, the frequentist would rely on Fisherian p-values or Pearsonian rejection limits or a mixture of these approaches, while the Bayesian would assign a total probability to the hypothesis. The underlying error or uncertainty estimates, however, are required to support both approaches. We think that a quantity for characterizing the error or uncertainty of a direct or indirect measurement which commits the user to either a frequentist or a Bayesian use of the measurements is of little use. Reference to Bayesianism alone cannot explain the claimed difference between ‘error’ and ‘uncertainty’.

¹¹We understand that subjective probability is related to the belief of a rational agent. Since two rational agents having access to the same information will believe the same, this variant of subjective probability should better be called ‘inter-subjective’ probability. This concept is often labelled ‘objective Bayesianism’.

950 The denial that a valid connotation of the term ‘error’ is a statistical characterization between a measured or estimated and the true value of the measurand would be an attempt to brush away centuries of scientific literature. This is, however, a matter of stipulation or convention and thus beyond the reach of a scientific argument. We thus take ~~GUM08-GUM-2008~~ to be conceding that both the concepts, error analysis and uncertainty assessment, aim at providing a statistical characteristic of
955 the imperfectness of a measurement or an estimate. We understand ~~GUM08-GUM-2008~~ in a sense that the problem of the error concept is that it conceives the estimated error as a statistical measure of the difference between the measured or estimated value and the true value. Since the true value is unknowable, according to ~~GUM08-GUM-2008~~ the term ‘error’ can neither be defined nor can its value be known.

960 It has been shown that the problem of the unknown true value of the measurand is a problem for the definition of terms like ‘error’ or ‘uncertainty’ only if the concept of an operational definition is ~~persued~~pursued. This concept, however, has its own problems and is by no means without alternative. As soon as the concept of an operational definition is given up, problems associated with defining the estimated error as a statistical estimate of the difference between the measurement or estimate
965 and the true value of the measurand disappear, and the problem remaining is only one of assigning a reasonable value to this now well-defined quantity.

Since ~~GUM08-GUM-2008~~ did not provide many reasons why, in the context of indirect measurements, the error allegedly cannot be estimated without knowledge of the true value, or why an uncertainty distribution does not tell us anything about the true value, we list the most obvious
970 ones one could put forward to bolster this claim. These are the problem of the base rate fallacy, the problem of non-linearity, and the problem that one can never know that the error budget is complete. The problem of the base rate fallacy can be solved by either performing a Bayesian inversion, or by conceiving the resulting distribution as a likelihood distribution. Astonishingly enough, the ~~GUM08~~GUM-2008’s “dispersion or range of values that could be reasonably attributed to the measurand” is determined without explicit consideration of prior probabilities and thus cannot be
975 interpreted in terms of posterior probability. The problem of nonlinearity can be solved ~~by the error scientist~~ either by assuming that the estimate is close enough to the true value and linearizing around this ~~poing point~~ or by Monte-Carlo-like studies. ~~The uncertainty scientist~~A GUM-oriented scientist, who has to avoid referring to the true value, is at a loss in the case of nonlinearity because any es-
980 timate of the uncertainty of the estimate will be correct only when evaluated at the true value or an approximation of it. The problem of the unknown completeness of the error budget can be tackled by performing comparisons between measurement systems. While this will never provide a positive proof of the completeness of the error budget, it still justifies rational belief in its completeness, and if error or uncertainty distributions are conceived as subjective probabilities in the sense of degrees of
985 rational belief, this is good enough. In summary, if (a) our reading of ~~GUM08-GUM-2008~~ is correct in the sense that the traditional error analysis can ~~connotate~~deal with a statistical quantity, and that

the key difference between the ‘error’ and ‘uncertainty’ concepts is their relation to the true value of the target quantity and (b), that our list of arguments against the error concept is complete, and finally, if (c) our refutation of these arguments is conclusive, then the claim that the ‘error’ concept and the ‘uncertainty’ concepts are fundamentally different is untenable¹¹.

Beyond this, reasons have been identified that put the applicability of the ~~GUM08-GUM-2008~~ concept to atmospheric measurements into question. At ~~the very~~ least we can state that ~~GUM08GUM-2008~~, by presenting their terminological stipulation about the terms ‘error’ and ‘uncertainty’ in the appearance of a factual statement, has triggered a linguistic discussion that distracted the attention from the more important issues how the principles of error or uncertainty estimation, whatever one prefers to call it, could be made better applicable to measurements beyond the idealized cases covered by their document.

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¹¹~~Building upon Willink and White (2012), we conclude instead, that ‘uncertainty’ and ‘uncertainty’ are (at least) two different things. This seems to hold at least when a frequentist and a Bayesian use this term. Ambiguities related to the term ‘error’ thus seem not to be removed but superseded by other ambiguities.~~

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